A gravity current of cold air traveling from NW *to* SE *in western Kansas, on a summer day in 1946. This picture was taken by Professor C. J. Posey, looking* SW, *while traveling west. Professor Posey kindly supplied this photograph, and furnished the following information: "The temperature was 96° F to* SE, *and 66° F to* NW. *Wind velocities were much higher inside the cold front than any we had noticed as it was approaching. Only the front edge was dusty. Later, the air cleared even though wind velocity stayed high."*

Stratified Flows

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PREFACE TO SECOND EDITION

"Stratified Flows" is the name I have given to this second edition of "Dynamics of Nonhomogeneous Fluids" for several reasons: The new title is shorter, the term *stratified flows* is now well established in the literature, and above all, usage has endowed the term with an association with gravity effects, in which the most interesting features of the dynamics of nonhomogeneous fluids reside.

I have taken the opportunity of this second edition to make corrections, omit some parts that now seem unimportant, and add new sections to the book. I have also added after each chapter critical notes on new results (available after 1965) that are closely related to the topics discussed in the main body of the book. These notes are not exhaustive, and to make amends for their sparseness I have provided at the end of the book an extensive bibliography, which has been enlarged and brought up to date. I hope that this bibliography will be useful to anyone seeking to inform himself about more recent developments in the dynamics of stratified flows. In other respects I have preserved the structure and points of view of the first edition.

I express my thanks to Mrs. Beverly Pyle for ably typing the manuscript for the second edition. Two friends, Y. C. Fung of La Jolla and Milton van Dyke of Palo Alto, have given me great encouragement in the course of my work on this edition. Professor van Dyke's unique combination of generosity and enthusiasm was just the nourishment I needed to complete this rather arduous task. Much of the revision of this book was done in 1978 during my sojourn in Karlsruhe, which was made possible by a Humboldt award. To the Alexander von Humboldt Foundation I express my sincere appreciation for giving me the time for undisturbed work.

Finally, I thank the staff at Academic Press for their splendid cooperation.

PREFACE TO FIRST EDITION

This book deals with the flow of a fluid of variable density or entropy in a gravitational field. Without gravity, the heterogeneity of a fluid can have only minor effects on its behavior. Indeed, the totality of solutions for the motion of a heterogeneous fluid can be shown to be completely equivalent to the totality of solutions for a homogeneous fluid, provided the flow is steady and the fluid inviscid and nondiffusive. Without heterogeneity, gravity has no effect whatsoever on the kinematics or dynamics of the flow, aside from contributing to the total pressure a hydrostatic part. Of course, since a free surface marks the boundary between a fluid and another of negligible inertia and viscosity, its presence implies the presence of heterogeneity, and, in fact, heterogeneity in an extreme form. If heterogeneity and gravity are both present, the situation is not merely more complicated. Often their interplay produces striking phenomena entirely unexpected.

Since gravity is omnipresent, and fluid homogeneity an exception rather than a rule, the subject dealt with in this book is relevant to most flows occurring in nature. If gravity effects have been totally ignored by aerodynamicists, it is because the airplane is too small and too fast for gravity effects to be appreciable. The magnificent development of aerodynamics since the turn of the century need not obscure the fact that there remain wide areas of fluid mechanics, no less challenging and rewarding, awaiting the energy of the scientific worker for their exploration. It is hoped that this little book will provide an introduction to one of these areas.

It is quite impossible to be exhaustive in the treatment, and, since this book is not intended to be a handbook, exhaustiveness is perhaps not even desirable. I have tried to use gravity as the warp and heterogeneity as the woof of a mat underlying and unifying the material presented. On it a few interlocking patterns are discernible. There is the pattern of particular history and its utilization (Chapter 1, Section 2; Chapter 3, Sections 1–14; and

Chapter 5, Section 4). There is the use of singularities and the inverse method for dealing with large-amplitude motions (most of Chapter 3 and a good part of Chapter 5). There is the pattern of eigenvalue problems (Chapters 2 and 4). In Chapter 2, the threads are provided by Sturm, Liouville, and Bôcher, whereas much more modern fabric is used in Chapter 4. From the point of view of content, Chapter 1 gives some general results serving as preliminaries to the following three chapters, which might be of some interest to workers in meteorology, oceanography, and engineering. Chapter 5 deals exclusively with seepage flow in porous media; it is hoped that the many recent results presented there, though far from being exhaustive, may be useful to civil engineers dealing with ground water flow and chemical engineers interested in oil seepage in the ground. Chapters 2 and 4 would be more useful to the oceanographer and the meteorologist if Coriolis effects of the earth's rotation had been discussed more thoroughly. The exclusion of a thorough discussion of these effects is not merely a matter of economy of space. I am afraid that such a discussion would distract from the main point of view, and I content myself with the provision of a separate chapter (Chapter 6), in which the analogy between the flow of a heterogeneous fluid in a gravitational field and the flow of an accelerating or rotating fluid is presented. This chapter is therefore a little rug on which the patterns of the main rug are traced out, in the hope of giving some satisfying sense of unity between the two categories of fluid flow.

Many familiar results in the flow of a heterogeneous fluid in a gravitational field are missing in this book. The most obvious ones are those on water waves and those on gravitational convection. There seems to be little need and even less possibility of including the extensive results on water waves presented in Lamb's and Professor Stoker's excellent books. Much of the familiar results on gravitational convection not mentioned in this book can be found in books on heat transfer. Although the analogy of the flow of a conducting fluid in a magnetic field to that of a stratified or rotating fluid is sometimes mentioned in Chapter 6, for full information on hydromagnetic stability the reader must be referred to the literature, and especially to Professor S. Chandrasekhar's excellent and extensive book. I have not included any detailed information on turbulence in a stratified fluid because it seems to me that the time for such an inclusion has not yet arrived.

The bibliography, like the subject matter, is not exhaustive. But if the reader looks up the references given in the papers and books referred to in this book, and repeats the process, it is unlikely that he will miss many important papers written before 1963. I have not read all the papers listed in the bibliography, but I thought a fairly extensive list might be useful.

It is my pleasant duty to thank the many people who have directly or indirectly contributed to this book. All those whose names are mentioned in the text have contributed to my understanding of the subject. I owe my initial interest in gravity effects on fluid flow to Professor Hunter Rouse, in hydrodynamics to Professor John S. McNown, and in hydrodynamic stability to Professor Chia-Chiao Lin. Dr. George K. Batchelor encouraged the writing of this book when it was first conceived at Cambridge, England, in 1960. Through both his work and his personal encouragement, Sir Geoffrey Taylor has been a constant source of inspiration. I wish to express my appreciation to Professors Otto Laporte, Louis N. Howard, and John W. Miles, who kindly read the manuscript and gave many valuable comments and suggestions, and to Dr. Walter R. Debler, who often participated in my research work with enthusiasm. I have been much encouraged in this work by Professor Yuan-Cheng Fung, a friend since 1934, and by Professor Thomas Farrell, a teacher of good writing to many of his friends. Almost all of my own research work in the field covered by this book has been sponsored by the Army Research Office (Durham) and by the National Science Foundation, which awarded me a senior post-doctoral fellowship in 1959–60 and a research grant in 1961. To both I am grateful. I also wish to thank Mrs. Jane Lamb and Miss Joanna Zaparyniuk for their patient and skillful typing of the manuscript. Finally, it is a great pleasure to express my sincere appreciation to The Macmillan Company for its painstaking and excellent work in putting the book in print and to record my special indebtedness to Mr. A. H. McLeod of Macmillan, for his patient and wonderful cooperation.

GUIDES FOR THE READER

Equation numbers are consecutively numbered only within each chapter. If an equation and a reference to it are in the same chapter, its number is used without further identification. If an equation in one chapter is referred to in the text of another chapter, its number is prefixed by the number of the chapter in which it appears. Thus, if Eq. (18) in Chapter 1 is referred to in Chapter 3, it is referred to as (1.18).

In the Bibliography, the italic number following the name of a journal is the volume number. The numbers (or number) following the volume numbers of a journal are page numbers (or number).

In a book covering such a wide range of subjects, it is difficult to have a system of symbols without sacrificing either the one-to-one correspondence or the customary use of some of the familiar symbols. I have tried to preserve the usage of familiar symbols and to achieve as much consistency and one-to-one correspondence as possible. When one symbol is used to denote more than one quantity, care has been taken to ensure that confusion is not likely to arise. I often define a symbol immediately after it is introduced, whether or not it has been defined before. Thus some symbols are defined several times. In this way I hope to prevent the annoyance readers often experience when obliged to turn to a list of symbols, or to take on a long journey to locate the place where a particular symbol is first introduced—sometimes even when a list of symbols is given, because the terse explanation in that list does not suffice. A list of symbols is not provided because I believe it will not be needed.

Chapter 1

PRELIMINARIES

1. GENERAL DISCUSSION OF THE EFFECTS OF DENSITY OR ENTROPY VARIATION

For an inviscid* fluid, the equations of motion are the Euler equations

$$\rho\left(\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}\right) = -\frac{\partial p}{\partial x_i} + \rho X_i, \tag{1}$$

in which t is the time, ρ is the density of the fluid, p is the pressure, X_i is the ith component of the body force per unit mass, u_i is the velocity component in the direction of x_i, with $i = 1$, 2, and 3, and x_1, x_2, and x_3 are Cartesian coordinates. In (1), repeated indices in one term indicate summation. Thus,

$$u_\alpha \frac{\partial u_i}{\partial x_\alpha} = u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3}.$$

This convention will be used throughout the book unless otherwise stated.

An examination of (1) reveals immediately that a nonuniformity in density will have two effects for an incompressible fluid. First, since density is a measure of the resistance of the fluid to acceleration, a nonuniformity in density is a nonuniformity in inertia, which will affect the flow except in the trivial case of steady parallel motion. Second, whenever a body force is present a nonuniformity in density is necessarily accompanied by a nonuniformity in body force per unit volume, which in general will affect the flow. The most important body force on earth is of course the gravitational force. In most flows of a nonhomogeneous fluid the inertia effect and the gravity effect of density variation are both present, and their interplay is the most intriguing aspect of such flows.

* If the fluid is a gas, inviscidness implies that the volume viscosity as well as the ordinary viscosity is zero.

If the fluid is compressible, the matter must be scrutinized with more care. Density may change as a direct consequence of a change in pressure, or an indirect consequence of a change in speed. But if the change of state of all the fluid particles is sufficiently slow and heat conduction is neglected, the entropy of each of the particles will remain constant,* and for each particle there is a unique relationship between the pressure and the density,

$$p = f(\rho). \tag{2}$$

If the entropies of all the particles are the same, this relationship is the same for the *entire* field of flow, and the flow is called homentropic. As the density varies in a homentropic flow from place to place at the same instant, and (in unsteady flows) from time to time at the same point, the pressure varies according to (2), so that the integral dp/ρ is an exact differential and $\int dp/\rho$ exists. Then (1) can be written as

$$\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} = -\frac{\partial}{\partial x_i}\int \frac{dp}{\rho} + X_i. \tag{3}$$

Thus the density ρ need not be associated with either the acceleration or the body force per unit mass, but can be absorbed in the term representing the pressure gradient, so that the *direct* inertia and gravity effects of density variation as discussed in the preceding paragraph are no longer manifest. This is not to say that the flow pattern will be unaffected by the density variation, because the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_\alpha)}{\partial x_\alpha} = 0 \tag{4}$$

has to be satisfied together with (2) and (3), and for high enough speeds the effect of density variation can be very pronounced. Indeed, a major part of ordinary aerodynamics is devoted to a study of Eqs. (2), (3), and (4), with the term X_i neglected. For large-scale flows in the atmosphere, the body-force (or gravity) term must be retained. But so long as (2) holds for the whole field of flow, gravity affects the flow only through the density, which is dependent upon it. As will be shown later in this book, if the flow is nonhomentropic, gravity has, *in addition* to this effect through the density, a far-reaching effect entirely absent in homentropic flows.

For the flow of a compressible fluid, it is therefore more meaningful to consider, not the effects of density variation, but the effects of entropy variation. For a gas with a constant ratio (γ) of the specific heats c_p and c_v,

$$\frac{\rho}{p^{1/\gamma}} = \text{constant} \cdot e^{-S/c_p}, \tag{5}$$

* The flow is then called isentropic.

so that $\rho/p^{1/\gamma}$ is a function of the entropy S alone. The effects of entropy variation can be brought out vividly by dividing (1) throughout by $p^{1/\gamma}$. The result is

$$\frac{\rho}{p^{1/\gamma}}\left(\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}\right) = -\frac{\partial}{\partial x_i}\left(\frac{\gamma}{\gamma - 1} p^{(\gamma-1)/\gamma}\right) + \frac{\rho}{p^{1/\gamma}} X_i. \tag{6}$$

Equation (6) as applied to a nonhomentropic gas can be compared with (1) as applied to a liquid of variable density. The quantity $\rho/p^{1/\gamma}$ in (6) corresponds to the density ρ in (1), and the effects of its nonhomogeneity in a gas are analogous to the effects of density variation in a liquid. We can thus speak of them as the inertia and gravity effects of entropy variation. In a gravitational field, nonhomentropy of a gas often has a far-reaching and pronounced effect on the flow, which is represented mathematically by the last term in (6).

2. THE INERTIA EFFECT OF DENSITY OR ENTROPY VARIATION

The components of acceleration of a fluid particle are given by

$$a_i = \frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}.$$

Since the temporal part $(\partial u_i/\partial t)$ is linear in u_i, whereas the convective part $(u_\alpha\, \partial u_i/\partial x_\alpha)$ is quadratic in the velocity components, there is no simple law to evaluate the effect of density (or entropy) variation on the velocity distribution. The discussion is profitable only if temporal and convective accelerations are discussed separately.

If the flow is unidirectional* (in the x_1-direction, say) so that the acceleration consists only of the temporal part, (1) can be written, with the last term neglected (because we are concerned at the moment with inertia effects of density variation),

$$\rho \frac{\partial u_1}{\partial t} = -\frac{\partial p}{\partial x_1}, \tag{7}$$

the other two equations contained in (1) being trivial, since $u_2 = u_3 = 0$. If the fluid is incompressible and, in addition, ρ is constant along the direction of flow,

$$\frac{\partial \rho}{\partial x_1} = 0. \tag{8}$$

Now the general equation of continuity is (4). The equation of incompressibility is, in general,

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_\alpha \frac{\partial \rho}{\partial x_\alpha} = 0, \tag{9}$$

* This situation can be realized by a wave maker in a confined channel at large distances from the wave maker. See Chapter 2, Section 15.

which states that the density of a fluid particle remains unchanged as the particle moves about. From (4) and (9) it follows that the equation of continuity for an incompressible fluid is

$$\frac{\partial u_\alpha}{\partial x_\alpha} = 0, \tag{10}$$

whether or not the flow is steady, and whether or not the fluid is homogeneous. In the special case discussed here, (10) becomes

$$\frac{\partial u_1}{\partial x_1} = 0, \tag{11}$$

and (9) becomes, by virtue of (8),

$$\frac{\partial \rho}{\partial t} = 0. \tag{12}$$

If ρ_0 is a reference density, and the new variables

$$u_1' = \frac{\rho}{\rho_0} u_1 \qquad \text{and} \qquad p' = p \tag{13}$$

are introduced, (12) permits (7) to be written as

$$\rho_0 \frac{\partial u_1'}{\partial t} = -\frac{\partial p'}{\partial x_1}, \tag{14}$$

and (8) permits (11) to be written as

$$\frac{\partial u_1'}{\partial x_1} = 0. \tag{15}$$

Thus (7) and (11) have been replaced by (14) and (15), which govern unidirectional flow of a homogeneous fluid of constant density ρ_0. Therefore, under the stated conditions, the effect of density variation on the velocity distribution is to make the velocity proportional to that of a homogeneous fluid subjected to comparable boundary conditions by the factor ρ_0/ρ. This is quite reasonable from the physical point of view.

Unfortunately, a transformation similar to that embodied in (13) cannot be found to reduce the equations governing unsteady flows of a nonhomentropic gas to those governing unsteady flows of a homentropic gas, even under the restriction of unidirectional flow and an equation like (8), with ρ replaced by $\rho p^{-1/\gamma}$.

If the flow is steady, so that the acceleration is purely convective, the inertia effect of density variation on the motion of an incompressible fluid can be evaluated by a rule different from but as simple as (13). With the body-force term neglected, (1) is now

$$\rho u_\alpha \frac{\partial u_i}{\partial x_\alpha} = -\frac{\partial p}{\partial x_i}. \tag{16}$$

The equation of continuity is still (10), but the equation of incompressibility now has the form

$$u_\alpha \frac{\partial \rho}{\partial x_\alpha} = 0. \tag{17}$$

If the new variables [Yih, 1958]

$$u_i' = \sqrt{\frac{\rho}{\rho_0}}\, u_i, \qquad p' = p \tag{18}$$

are introduced, (16) and (10) can be written, by virtue of (17), as

$$\rho_0 u_\alpha' \frac{\partial u_i'}{\partial x_\alpha} = -\frac{\partial p'}{\partial x_i}, \tag{19}$$

and

$$\frac{\partial u_\alpha'}{\partial x_\alpha} = 0. \tag{20}$$

But (19) and (20) govern the flow of a homogeneous fluid of constant density. Therefore it can be stated that, with gravity effects neglected, to every flow (called associated flow for convenience) of a homogeneous fluid of constant density, rotational or irrotational, correspond infinitely many stratified flows which are related to the associated flow (defined by u_i' and p') through equations (18). Thus, with gravity effect neglected, every known flow in classical hydrodynamics represents a class of stratified flows with the same flow pattern but different velocities.

A similar development for a compressible fluid with variable entropy is possible if its flow is steady, and if the change of state of each fluid particle is isentropic. Isentropy presumes a gradual change of state without heat conduction, and is expressed mathematically by the equation

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + u_\alpha \frac{\partial S}{\partial x_\alpha} = 0, \tag{21}$$

in which S is the entropy. If the specific heats of an ideal gas are constant, $\rho/p^{1/\gamma}$ is a function of S alone, as stated in (5). Hence isentropy can also be expressed by

$$\left(\frac{\partial}{\partial t} + u_\alpha \frac{\partial}{\partial x_\alpha}\right)\left(\frac{\rho}{p^{1/\gamma}}\right) = 0. \tag{22}$$

For steady flows, the equation of isentropy is

$$u_\alpha \frac{\partial}{\partial x_\alpha}\left(\frac{\rho}{p^{1/\gamma}}\right) = 0. \tag{23}$$

Now the equations of motion are still (16), with the gravity term neglected. The equation of continuity is now, for steady flows,

$$\frac{\partial(\rho u_\alpha)}{\partial x_\alpha} = 0. \tag{24}$$

As a consequence of (23), the quantity λ defined by

$$\lambda = \frac{\rho}{\rho_0}\left(\frac{p_0}{p}\right)^{1/\gamma} = \text{constant} \cdot e^{-S/c_p}, \tag{25}$$

in which ρ_0 is a reference density and p_0 a reference pressure, satisfies the equation

$$u_\alpha \frac{\partial \lambda}{\partial x_\alpha} = 0$$

if the flow is steady. Indeed, under the assumption of steadiness, the more general equation

$$u_\alpha \frac{\partial F(\lambda)}{\partial x_\alpha} = 0 \tag{26}$$

is true, in which $F(\lambda)$ is an arbitrary function of λ.

By virtue of (26), the transformation [Yih, 1960c]

$$u_i' = \sqrt{\lambda} u_i, \qquad \rho' = \rho/\lambda, \qquad p' = p \tag{27}$$

reduces (16) and (24) to

$$\rho' u_\alpha' \frac{\partial u_i'}{\partial x_\alpha} = -\frac{\partial p'}{\partial x_i}, \tag{28}$$

and

$$\frac{\partial(\rho' u_\alpha')}{\partial x_\alpha} = 0, \tag{29}$$

in which, as can be readily verified,

$$\frac{\rho'}{p'^{(1/\gamma)}} = \frac{\rho_0}{p_0^{1/\gamma}} = \text{constant}. \tag{30}$$

Thus (27) transforms the equations governing the flow of a nonhomentropic gas to those governing the flow of a homentropic gas. In other words, to every homentropic flow, rotational or irrotational, correspond infinitely many nonhomentropic flows which are related to it through (27), and every known homentropic flow represents a class of known nonhomentropic flows.

In this section inertia effect has been discussed, and in the discussion the term representing gravity in (1) has been neglected. If the flow is exactly horizontal, the only effect of gravity is to make the pressure hydrostatic in the direction of the vertical. Thus if the associated flow is horizontal (for example, for flow past a vertical cylinder) the corresponding flow of a stratified fluid, also horizontal, satisfies the equations of motion exactly, even in the presence of a gravitational field. This statement is true if the flow is steady, whether the fluid is incompressible or compressible.

The conclusions reached for steady flows through transformation (18) for liquids and (27) for gases are valid even if the variation of density or entropy across the streamlines is not continuous. What makes the transformations

(18) and (27) successful is (17) and (23), both of which are equations of particular history.

Since the success of (18) and (27) depends on the conservation of entropy along each streamline, and since entropy is not conserved at a shock wave, it seems at first sight that the existence of an associated flow for any steady flow of a nonhomentropic gas breaks down if a shock wave is present, and the flow is to be considered in its entirety, not merely as a collection of separate regions. Closer examination of the shock conditions reveals the surprising fact that this is not so.

Only the case of two-dimensional flow will be considered here, because the three-dimensional case differs from it only in complexity, not in principle. With u and v denoting velocity components in the directions of increasing x and y, respectively, and with β denoting the local angle of inclination of the shock wave, continuity of mass flow across the shock demands [Liepmann and Puckett, 1947, p. 51]

$$\rho_1 u_1 \sin\beta = \rho_2(u_2 \sin\beta - v_2 \cos\beta),$$

in which ρ is the density, and the subscripts 1 and 2 indicate preshock and postshock conditions, respectively. The preshock flow is assumed parallel to the x-axis. Hence $v_1 = 0$. The equation for the momentum normal to the shock wave is

$$p_1 + \rho_1 u_1^2 \sin^2\beta = p_2 + \rho_2(u_2 \sin\beta - v_2 \cos\beta)^2,$$

in which p is the pressure, whereas the conservation of momentum parallel to the shock wave demands

$$\rho_1 u_1^2 \sin\beta \cos\beta = \rho_2(u_2 \sin\beta - v_2 \cos\beta)(u_2 \cos\beta + v_2 \sin\beta).$$

The energy equation remains

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{1}{2}(u_2^2 + v_2^2) + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}.$$

If we insist on making the transformation (27), and determine λ in (27) from the preshock conditions, the associated flow is parallel and homentropic before the shock, and is irrotational if λu_1^2 is constant throughout. The postshock flow is not homentropic even for the associated flow, and is governed by the equations

$$\left(u'\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right)(u', v') = -\frac{1}{\rho'}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)p',$$

$$\frac{\partial(\rho' u')}{\partial x} + \frac{\partial(\rho' v')}{\partial y} = 0,$$

$$\left(u'\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right)\left(\frac{p'}{\rho'^{\gamma}}\right) = 0,$$

because λ is constant on each streamline by choice. If the primes in these equations are dropped, they are precisely the equations governing the post-shock flow. Furthermore, the equations of continuity, momenta, and energy at the shock are still satisfied if all the quantities except β and γ are primed. Thus, the transformation (27) preserves the pattern of the *entire* flow, and only makes the preshock flow homentropic. In other words, even when a shock is present, to every flow which is homentropic before the shock correspond infinitely many flows which have entropy stratification before the shock, but which have exactly the same *pattern for the entire field of flow.*

It may be interesting to record the process by which the transformation (18) was discovered. It is well known that in steady flows of an incompressible fluid the Bernoulli equation is valid along a streamline and results from the integration of (1) along the direction of flow, provided the body force possesses a potential Ω. This equation is

$$\frac{\rho u_\alpha u_\alpha}{2} + p + \rho\Omega = C \tag{31}$$

in which C is a constant along a streamline, but may vary from streamline to streamline. With gravity effects neglected, the Bernoulli equation becomes

$$\frac{\rho u_\alpha u_\alpha}{2} + p = C. \tag{32}$$

Suppose there are two homogeneous liquids (of density ρ_1 and ρ_2) in contact. The Bernoulli equations applied at the interface to the two fluids in turn are

$$\left(\frac{\rho u_\alpha u_\alpha}{2}\right)_1 + p_1 = C_1 \tag{33}$$

and

$$\left(\frac{\rho u_\alpha u_\alpha}{2}\right)_2 + p_2 = C_2 . \tag{34}$$

Since $p_1 = p_2$ at the interface, these equations can be true at every point of the interface only if

$$(\rho u_\alpha u_\alpha)_1 = (\rho u_\alpha u_\alpha)_2 , \tag{35}$$

except in trivial cases. Now if the directions of the velocities of the two fluids at the interface are identical, (35) reduces to

$$(\sqrt{\rho}\, u_i)_1 = (\sqrt{\rho}\, u_i)_2 ,$$

which leads immediately to the velocity transformation in (18). Indeed, the flow pattern of a homogeneous liquid can be maintained by a liquid with discontinuous density only if the velocity is connected with that of the associated flow (of the homogeneous fluid) through (18). Otherwise the pressure cannot be balanced at the interface. As the discontinuities are indefinitely reduced, the same conclusion is reached in the limit for a liquid with continuous density variation. After the success of (18), the transformation (27) was obtained by analogy, and its validity proved by a similar development.

If the associated flow defined by the transformations (18) or (27) satisfies the kinematic boundary conditions at some boundaries, it is evident that the actual flow does also, since the velocity of one flow at any point differs from that of the other flow at the same point only by a scalar factor. Furthermore, if gravity effects are neglected, satisfaction of a dynamic condition at a boundary (which in steady flows must be a streamline) by an associated flow guarantees satisfaction of the same condition by the actual flow, because velocity appears in the combination

$$\rho u_\alpha u_\alpha/2 \qquad \text{or} \qquad \lambda u_\alpha u_\alpha/2$$

in the Bernoulli equation governing pressure variation along the boundary. See (32) for the case of an incompressible fluid. There is a corresponding one for a compressible fluid. (For two-dimensional steady flows, see the form of H for an incompressible fluid in (9) of Chapter 3, and for a compressible fluid in (91) of that chapter. The Bernoulli equation states that H is a constant along a streamline in steady flow. The forms of H for three-dimensional flows differ from those in (3.9) and (3.91) only in that the square of the speed contains a third term for these flows.)

3. THE GRAVITY EFFECT OF DENSITY OR ENTROPY VARIATION

In slow steady flows, the gravity effect attending density stratification dominates the inertia effect, and is often very striking. For definiteness, consider the flow due to a body moving with a small and constant velocity in a fluid with the density gradient in the x_3-direction when undisturbed. The flow can be made steady by the use of a frame of reference moving with the body. In ordinary Cartesian notation, Eqs. (1) become, for steady motion,

$$\rho\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u, v, w) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)p + (0, 0, -g\rho). \quad (36)$$

The equation of incompressibility is

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0. \quad (37)$$

In (37),

$$\frac{\partial \rho}{\partial z} = O(1). \quad (38)$$

Elimination of p by cross differentiation of the first and third equations in (36) produces

$$\frac{\partial \rho}{\partial x} = O(\varepsilon^2), \quad (39)$$

in which ε denoted the order of magnitude of u, v, or w. Similarly, elimination of p between the second and third equations in (36) produces

$$\frac{\partial \rho}{\partial y} = O(\varepsilon^2). \tag{40}$$

Thus (37), (38), (39), and (40) yield

$$w = O(\varepsilon^3). \tag{41}$$

With an error of the fourth order in ε, then, (36) can be written as

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}, \tag{42}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}, \tag{43}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g. \tag{44}$$

The equation of continuity can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{45}$$

with an error to the third order in ε. Equation (44) states that the pressure distribution is hydrostatic in the z-direction, and (42) and (43) are precisely the Euler equations for a horizontally homogeneous fluid in horizontal motion. At any given elevation, the fluid behaves like a homogeneous fluid, and as a consequence the vanishing of

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

persists in the fluid if ζ is zero upstream. In the case under discussion, ζ is zero upstream, where the velocity is constant. Therefore the flow is horizontally irrotational, and can be obtained by superposing infinitely many (two-dimensional!) irrotational flows of infinitesimally thin fluid layers, one on top of the other. Of course, for this result to be meaningful, the body in motion must be smooth. But it can be any (smooth) three-dimensional body.* The velocity at infinity is uniform.

The result obtained above means that whenever a body is placed in a stratified fluid moving slowly, the fluid will go around the body horizontally. The horizontality of the motion can be easily verified by anyone interested enough to make a pan of stratified fluid with salt and to move a spoon through it slowly, using some ink drops as tracers. If the body is a cylinder with a

* See, however, the last paragraph of Section 16 of Chapter 2.

horizontal axis perpendicular to the direction of flow, and either is infinitely long or terminates at vertical walls, the fluid cannot go around it, and intuitively one would expect the fluid contained between two horizontal planes just touching the cylinder from above and from below to be stationary relative to the cylinder. That this is indeed so (if viscosity is neglected) can be seen from the fact that w is zero approximately and v is zero exactly. Thus (45) assumes the form [Yih, 1959e]

$$\frac{\partial u}{\partial x} = 0,$$

which states that the fluid before and after the cylinder will attach to the cylinder like a solid body. This prediction has been essentially verified by experiments [Yih, 1959d], though the velocity distribution in the fluid train was found nonuniform, as a consequence of viscosity.

4. CREATION OF VORTICITY BY NONHOMOGENEITY

Except in trivial or a few very special cases, vorticity is created in a nonhomogeneous fluid in motion. The simplest way to show this is by cross differentiation of (1), after division by ρ, and setting i equal to 1 and 2 in turn. If the difference of the results of the cross differentiation is taken, and ξ_3 is used to denote the vorticity component

$$\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},$$

we have, after some simplification,

$$\frac{D\xi_3}{Dt} + \xi_3 \frac{\partial u_\alpha}{\partial x_\alpha} = \xi_\alpha \frac{\partial u_3}{\partial x_\alpha} - \left(\frac{\partial \rho^{-1}}{\partial x_1}\frac{\partial p}{\partial x_2} - \frac{\partial \rho^{-1}}{\partial x_2}\frac{\partial p}{\partial x_1}\right). \tag{46}$$

Since

$$\frac{\partial u_\alpha}{\partial x_\alpha} = -\frac{1}{\rho}\frac{D\rho}{Dt},$$

(46) can be written as

$$\frac{D}{Dt}\left(\frac{\xi_3}{\rho}\right) = \frac{\xi_\alpha}{\rho}\frac{\partial u_3}{\partial x_\alpha} + \frac{1}{\rho^3}\left(\frac{\partial \rho}{\partial x_1}\frac{\partial p}{\partial x_2} - \frac{\partial \rho}{\partial x_2}\frac{\partial p}{\partial x_1}\right). \tag{47}$$

By symmetry the vorticity equations can be written in the vector form

$$\frac{D}{Dt}\frac{\boldsymbol{\xi}}{\rho} = \left(\frac{\boldsymbol{\xi}}{\rho}\cdot \mathbf{grad}\right)\mathbf{u} + \frac{1}{\rho^3}(\mathbf{grad}\,\rho \times \mathbf{grad}\,p). \tag{48}$$

In the special case of an incompressible fluid, (46) becomes

$$\frac{D\xi_3}{Dt} = \xi_\alpha \frac{\partial u_3}{\partial x} + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x_1}\frac{\partial p}{\partial x_2} - \frac{\partial \rho}{\partial x_2}\frac{\partial p}{\partial x_1}\right), \tag{49}$$

so that the vector form of the vorticity equations is

$$\frac{D}{Dt}\xi = (\xi \cdot \mathbf{grad})\,\mathbf{u} + \frac{1}{\rho^2}(\mathbf{grad}\,\rho \times \mathbf{grad}\,p). \tag{50}$$

The left-hand side of (50) represents the rate of change of vorticity. The first term on the right-hand side represents the rate of change of ξ due to stretching and turning of the vortex lines. Thus the last term in (50) represents the creation of vorticity by density variation. The same interpretation applies to the last term in (48), except that ξ/ρ must replace ξ in the argument, because of the effect of compressibility. Significant is the fact that vorticity is created only if the gradient of ρ and that of p are not in the same direction. If ρ is constant, of course the term responsible for the creation of vorticity vanishes, and persistence of irrotationality follows from the Helmholtz-Kelvin theorem of the preservation of circulation along any material circuit and from the well-known kinematic relationship between circulation and vorticity. If the fluid is homentropic, so that ρ is a function of p only over the whole field of flow, **grad** ρ and **grad** p are in the same direction, and the last term in (48) vanishes. Thus again the persistence of irrotationality follows from the Hankel-Kelvin theorem of preservation of circulation in barotropic flows. A barotropic flow is one for which a relation $f(p, \rho, t)$ exists. See Truesdell [1954, p. 92].

Since p is affected by the inertia and gravity terms in (1), the last term in (48) or (50) already embodies the inertia and gravity effects of entropy or density variation.

The creation of vorticity entails the creation of circulation. Let an elemental length along a material circuit C be ds, with components dx_1, dx_2, and dx_3, and let the difference of the velocity component u_i at the ends of ds be du_i. Then

$$du_i = \frac{\partial u_i}{\partial x_\alpha}\,dx_\alpha = \frac{D}{Dt}\,dx_i. \tag{51}$$

Multiplying (1) by dx_i/ρ, summing over i, and integrating over C, we have

$$\int_C dx_i\,\frac{Du_i}{Dt} = \int_C X_i\,dx_i - \int_C \frac{1}{\rho}\frac{\partial p}{\partial x_i}\,dx_i, \tag{52}$$

or, by virtue of (51),

$$\frac{D}{Dt}\int_C u_i\,dx_i = \int_C (X_i\,dx_i + u_i\,du_i) - \int_C \frac{1}{\rho}\frac{\partial p}{\partial x_i}\,dx_i. \tag{53}$$

If the body force X_i has a single-valued potential, and u_i is single-valued, (53) becomes

$$\frac{D\Gamma}{Dt} = -\int_C \frac{1}{\rho}\frac{\partial p}{\partial x_i}\,dx_i, \tag{54}$$

in which

$$\Gamma = \int_C u_i \, dx_i \tag{55}$$

is the circulation. Now, by Stokes' theorem, with S indicating a surface bounded by C,

$$\int_C \frac{1}{\rho}\frac{\partial p}{\partial x_i}\, dx_i = \iint_S \mathbf{curl}\left(\frac{1}{\rho}\,\mathbf{grad}\; p\right)\cdot d\mathbf{A}$$

$$= \iint_S \left(\mathbf{grad}\,\frac{1}{\rho} \times \mathbf{grad}\; p\right)\cdot d\mathbf{A}, \tag{56}$$

since **curl grad** $p = 0$. Thus

$$\frac{D\Gamma}{Dt} = -\iint \left(\mathbf{grad}\,\frac{1}{\rho} \times \mathbf{grad}\; p\right)\cdot d\mathbf{A}. \tag{57}$$

This is the theorem of V. Bjerknes.

The creation of vorticity and circulation, apart from that resulting from viscosity, is thus a special feature of flows of a fluid with nonhomogeneous density or entropy. This feature sometimes affects the character of the flow in a remarkable manner, as was indicated in Section 3 when the creation of vortex sheets (in an inviscid fluid) by the slow translation of an immersed cylinder was discussed, and as will be demonstrated further in Chapter 3.

5. THE STRUCTURE OF STRATIFIED FLOWS

If the closed curve C is taken on a surface of constant density or entropy, (54) can be written as (the integral being evaluated at any instant)

$$\frac{D\Gamma}{Dt} = -\int_C \frac{dp}{\rho} = 0, \tag{58}$$

since the integrand is an exact differential and the pressure and the density are single-valued. This equation will continue to be valid as time goes on, because either the density or the entropy is assumed to be conserved on the material circuit C. Equation (58) embodies the

Circulation Theorem. *The circulation around a closed material circuit lying in a surface of constant density or entropy is time-independent for an inviscid and nondiffusive nonhomogeneous fluid.*

In particular, if Γ is zero around C, it will continue to be zero. For a stratified flow started from rest, the circulation around any closed material circuit in a surface of constant density or entropy remains zero. From Stokes' theorem on circulation and vorticity then follows the

Vorticity Theorem. *For a stratified flow started from rest, the vortex lines are always imbedded in a surface of constant density or entropy, provided the fluid is inviscid and nondiffusive.*

Such a flow is called layerwise irrotational, because it is irrotational in a surface of constant density or entropy.

In the presence of nonhomogeneity a vortex line no longer moves with the fluid. But the theorem just reached indicates that the constant-density or constant-entropy surfaces are also a family of vorticity surfaces, and this family moves with the fluid. This situation is further elucidated by (48) and (50). The terms containing

$$\mathbf{grad}\,\rho \times \mathbf{grad}\,p$$

in (48) and (50) are responsible for the creation of vorticity, and the vector of the vorticity so created will be in the constant-density or constant-entropy* surfaces. But for this term the vortex lines would move with the fluid. The presence of this term merely has the effect of turning the vortex lines in the isopycnic or homentropic surfaces from the positions they would otherwise occupy.

For steady flows, isopycnic or homentropic surfaces are stream surfaces. If such flows have been started from rest, the vorticity lines will be embedded in such surfaces, which then contain not only streamlines but also vortex lines. If $\mathbf{v}$ represents the velocity vector and $\boldsymbol{\xi}$ the vorticity vector, $\mathbf{v} \times \boldsymbol{\xi}$ is then normal to the surfaces of constant ρ or entropy S. For steady flows, Eq. (3) can be written in the vector form

$$\mathbf{grad}\left(\frac{q^2}{2} + \Omega\right) + \frac{1}{\rho}\,\mathbf{grad}\,p - \mathbf{v} \times \boldsymbol{\xi} = 0, \tag{59}$$

in which q is the speed. In a constant-density or constant-entropy surface, let the curvilinear orthogonal coordinates α and β be introduced. The density or entropy does not change with α or β. Since $\mathbf{v} \times \boldsymbol{\xi}$ is normal to the α–β plane, and *in the α–β plane*

$$\frac{1}{\rho}\,\mathbf{grad}\,p = \mathbf{grad}\int \frac{dp}{\rho},$$

(8) can be integrated with respect to α and β to produce the

Bernoulli Theorem.

$$\frac{q^2}{2} + \int \frac{dp}{\rho} + \Omega = F(\rho) \quad \textit{or} \quad F(S). \tag{60}$$

This theorem is valid all over the field. The function $F(\rho)$ or $F(S)$ can be determined from data on a line piercing through all constant-density or constant-entropy surfaces.

* This can be seen by writing

$$\mathbf{grad}\,\rho \times \mathbf{grad}\,p = p^{1/\gamma}\,\mathbf{grad}\,\frac{\rho}{p^{1/\gamma}} \times \mathbf{grad}\,p.$$

NOTES

Section 2

1. Munk and Prim [1947] have demonstrated the possibility of constructing steady isentropic flows of a gas having the same pattern of streamlines but with the entropy or stagnation enthalpy varying from streamline to streamline. This has also been expounded clearly by Tsien [1958]. The transformation I have given in Section 2 makes more transparent, I think, the correspondence between a homentropic flow and nonhomentropic flows of the same flow pattern and the possibility of even including shocks in the flows without destroying the correspondence.

Section 3

1. Bretherton [1967] investigated the time-dependent motion due to a cylinder moving in an unbounded rotating fluid. The axis of the cylinder is perpendicular to the axis of rotation of the fluid, and the cylinder moves in the direction of the axis of rotation. Nonlinear terms of the acceleration are neglected, and Bretherton found Taylor columns, both before and behind the cylinder, growing indefinitely in length with a finite speed.

Bretherton claimed in the abstract of his paper that "an identical analysis may be applied with a similar interpretation in terms of internal gravity waves rather than inertial waves." The analogy would be a close one if Bretherton had considered a sphere moving along the axis of rotation instead of a cylinder, since in the case of the sphere there would be lee waves similar to the lee waves behind a cylinder moving horizontally in a stratified fluid. These lee waves may not be prominent if the effects of viscosity and (in the case of a stratified fluid) diffusivity are sufficiently great. See, for instance, Fig. 72 in Chapter 6 and Freund and Meyer [1972]. The lack of close physical analogy of Bretherton's problem with the column Yih [1959e] mentioned in Section 3 requires a separate treatment of the stratified-flow problem.

2. The phenomenon described by Yih [1959d] occurs when the Froude number is very small but the Reynolds number large. In the simple demonstration given in Section 3 both the effect of viscosity and the effect of diffusivity are ignored, and it is then found that u is independent of the horizontal coordinate in two-dimensional flows, so that there is a column ahead of as well as behind the slowly moving body causing the flow. Janowitz [1968], in an investigation of this phenomenon using Oseen's approximation and assuming a concentrated body force at the origin representing the drag, included the effect of viscosity, but ignored the effect of diffusivity, and found an upstream wake in the form of a column of nearly stagnant fluid, in agreement with the asymptotic solution of Long [1959] for the far-upstream region, and lee waves on the downstream side. His results are for any Reynolds number, with graphical results given for Reynolds number equal to 1, 10, and 100. Later Janowitz [1971] considered the slow motion of a flat plate broad-side on in a stratified fluid. But Freund and Meyer [1972] found that the momentum integrals far upstream in Janowitz's Eq. (4) diverge and argued somewhat vaguely that since the momentum integral far downstream cannot balance the upstream momentum integral, because there is no downstream wake, the drag for Janowitz's solution [1968] is infinite. While the arguments leading to this conclusion still lack rigor and precision, the divergence of the Janowitz's momentum integral far upstream does cause concern. In this regard it is noteworthy that neither Janowitz [1968] nor Freund and Meyer [1972] realized that in the momentum integral the normal stress must be used instead of the pressure, so that a term $2\mu\ \partial u/\partial x$ (where μ is the viscosity, u the horizontal velocity, and x the horizontal Cartesian coordinate) is lacking in Janowitz's so-called momentum integrals. This point makes the definition of drag in Janowitz's Eq. (4) erroneous and the dispute over whether the drag is infinite for his solution premature. It is possible that the missing term I have just pointed out is negligible, but this is unlikely in view of the fact that the terms

$$p_x - (u_{xx} + u_{yy}),$$

(p and u being now *dimensionless* pressure and horizontal velocity and y the vertical Cartesian coordinate) are included in Janowitz's Eq. (6a), implying that p and u_x are of the same order of magnitude. It should be noted also that Janowitz's Eq. (4) defining the drag can be corrected without affecting his subsequent developments. Hence his oversight does not have more consequence than for the convergence or divergence of the corrected momentum integrals.

Graebel [1969] considered the same problem as Janowitz [1968] without representing the effect of the solid body by a concentrated body force, but he ignored inertial effects entirely. He used stretched (or rather compressed) coordinates and gave an asymptotic solution far upstream for the upstream wake, which agrees with Long's solution [1959], and a downstream solution for a body with a circular shape on the downstream side. The downstream solution exhibits lee waves attenuating as the distance downstream increases, as in Janowitz's solution. Graebel argued that the shape of the upstream side of the body is unimportant when the (internal) Froude number is low. His conclusion that the drag is independent of both the viscosity and the speed may seem puzzling at first, but it is quite reasonable since, as he argues, the viscous drag is small because the speed is small but the viscosity not necessarily large, so that gravity forces dominate forces induced by viscosity.

Graebel, by restricting attention to very small Reynolds numbers and avoiding the conventional heavy machinery of Fourier transforms used by Janowitz, was able to give simpler and more transparent results than Janowitz's.

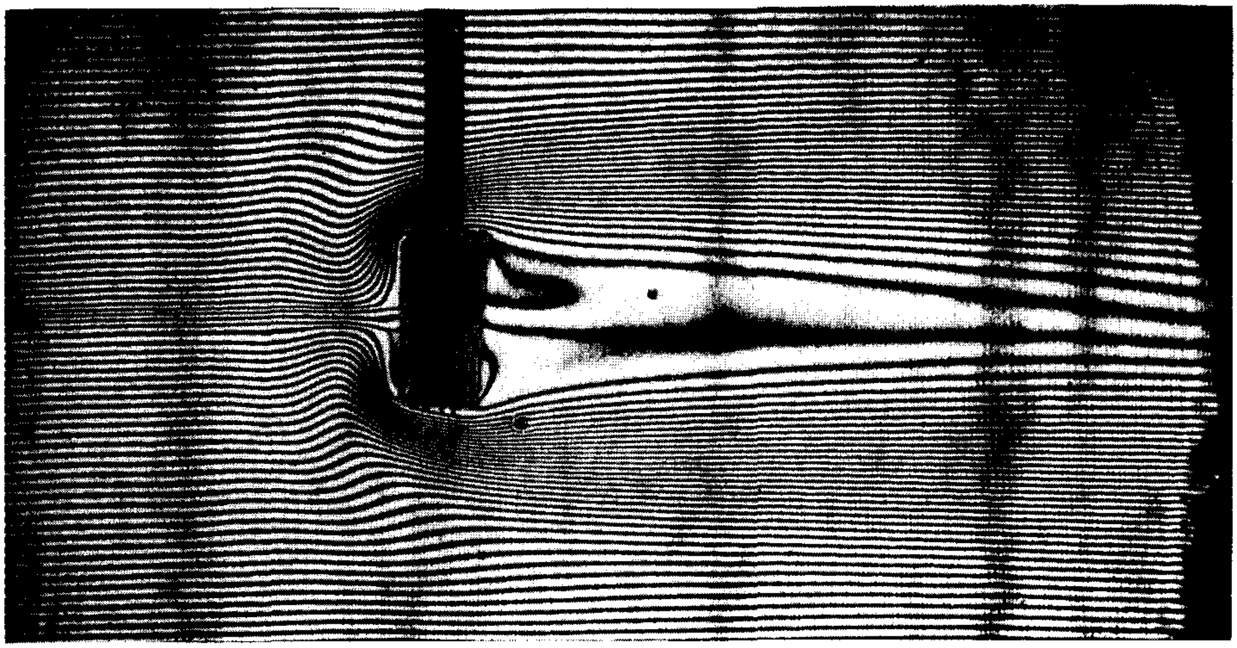


FIGURE N1.1. Interferogram of a flow with a density gradient of 3.3×10^{-4} gm/cm^4 and a towing speed of 0.05 cm/sec to the right. The Reynolds number R based on height of the rectangular cross section of the bar is 6.78. The internal Froude number $F = 4.8 \times 10^{-3}$. (*Courtesy of the Royal Society of London and of Professors W. R. Debler and C. M. Vest* [*see their work of 1977*]).

Both Janowitz and Graebel ignored diffusive effects. Freund and Meyer [1972], like Graebel, ignored inertial effects entirely, and they took the pressure distribution as hydrostatic. They considered a diffusion equation that took the density gradient of the undisturbed fluid into account in the convective terms. As a result of this measure they obtained, in effect, a final partial differential equation in which the highest order of the derivatives is 6 rather than 4, as for the differential system treated by Janowitz and Graebel. Consequently, at very low speeds Freund and Meyer obtained both upstream and downstream wakes.

More elaborate experiments than Yih's [1959d] have been performed by Barnard and Pritchard [1975], who used schlieren pictures, Debler and Vest [1977], who used a laser interferometer to study flows at internal Froude numbers of the order of 10^{-3} and Reynolds number of the orders of 1 and 10, and Debler [1978]. When the speed is low an upstream wake was always found, and depending on the importance of diffusive effects (i.e., on the Péclet number), there may or may not be a downstream wake. Figure N1.1 shows an upstream wake and lee waves, whereas Fig. N1.2 shows both upstream and downstream wakes.

Section 4

1. In an attempt to study the generation of streamwise vorticity in a steady flow of a stratified fluid, Marris [1964] took into account the inertial effect of density variation by using the transformation (18), but, curiously

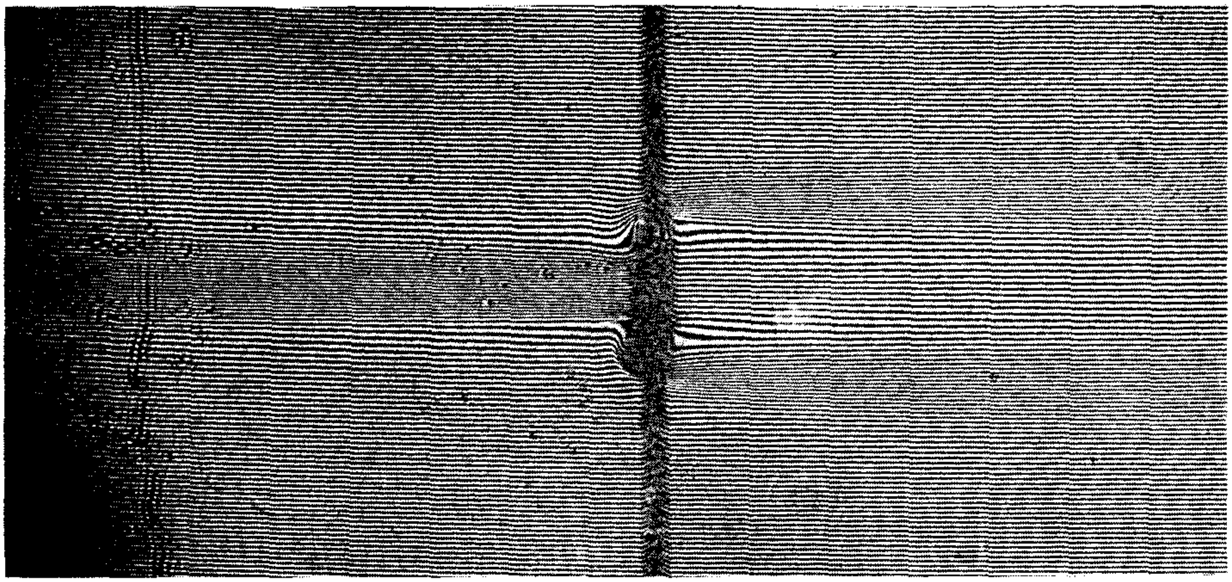


FIGURE N1.2. Interferogram of a flow with a density gradient of 5.64×10^{-4} gm/cm^4 and a towing speed of 3.22×10^{-3} cm/sec to the right. $R = 3.22 \times 10^{-2}$ and $F = 4.33 \times 10^{-3}$. (*Courtesy of Professor W. R. Debler.*) Another interferogram obtained by Debler is for a density gradient of 8.5×10^{-4} gm/cm^4, a towing speed of 6×10^{-4} cm/sec, bar height of 1 cm, $R = 0.6$, and $F = 6.6 \times 10^{-4}$. That photograph shows even greater symmetry of upstream and downstream wakes but has such fine fringes that their reproduction is difficult. It is now beyond doubt that at lower and lower speeds the column of fluid moving with the towed bar becomes more and more symmetric with respect to the bar.

enough, ignored the gravity effect of density variation, since he assumed the body force per unit volume to have a potential. This procedure is exactly the opposite of the Boussinesq approximation and is not applicable when gravity effects are important. At high speeds, when Marris' approximation can be applied, the transformation (18), of course, reduces the problem to that for a homogeneous fluid, for which the results concerning vorticity are well known.

Chapter 2

WAVES OF SMALL AMPLITUDE

1. INTRODUCTION

The phenomenon of waves or surges in a nonhomogeneous fluid is a clear example illustrating the essential roles played by density or entropy stratification on the one hand, and by gravity on the other, in its production. Indeed, gravity waves in a homogeneous fluid are impossible without a free surface, which is really a surface of density discontinuity—an extreme case of density stratification.

The physical cause of the oscillatory nature of all waves is the restoring force acting on a material particle as the particle is displaced from its mean position. If the restoring force is gravitational in nature, then density or entropy variation is essential for the restoring force to exist. This situation can be clarified by a consideration of static stability.

If the fluid is incompressible, then a fluid particle of density ρ_1 will retain its density at the new position where it has been (or is to be) displaced. For definiteness let the new position be lower than the old, and let the density there be ρ_2. If ρ_2 is greater than ρ_1, the weight of the particle is more than balanced by the total force resulting from hydrostatic pressure acting on the surface of the particle, and the net force on the particle is a restoring force. The fluid (or the stratification) is therefore stable. If ρ_2 is less than ρ_1, the opposite is true, the force is an aggravating force, and the fluid is unstable. Oscillation, which characterizes wave motion, is therefore possible only if the stratification is statically stable—that is, if the density decreases upward.

If the fluid is compressible, the density of a fluid particle will not remain constant when displaced to a new location of a different pressure. If heat conduction is neglected and the transition from the old position to the new is effected gradually, so that the change of state of the particle is isentropic, then the new density $\tilde{\rho}_1$ of the particle can be computed from

$$\frac{\tilde{\rho}_1}{p_2^{1/\gamma}} = \frac{\rho_1}{p_1^{1/\gamma}}, \tag{1}$$

since the pressure within the particle must be the same as the prevalent pressure p_2. Now if the ambient entropy at the old position is more than that at the new (at a lower elevation), then, according to (5) of Chapter 1,

$$\frac{\rho_1}{p_1^{1/\gamma}} < \frac{\rho_2}{p_2^{1/\gamma}}, \tag{2}$$

and (1) and (2) produce

$$\tilde{\rho}_1 < \rho_2, \tag{3}$$

so that the weight of the fluid particle is more than balanced by the total force arising from the hydrostatic pressure acting on the surface of the particle, and the net force is a restoring force. The fluid is therefore stable. If the entropy decreases with elevation the opposite is true, and the fluid is unstable. Wave motion can only exist in a stably stratified fluid, as will be shown later.

The concept of potential density is a very useful one when considering the behavior of a compressible fluid. The potential density of a compressible fluid or a portion of it is its density when its pressure is brought isentropically to a fixed value. Thus a compressible fluid at rest is stable if its potential density decreases upward.

2. THE GOVERNING DIFFERENTIAL EQUATIONS

The differential system governing wave motion of a compressible fluid with variable entropy will first be derived. That for an incompressible fluid can then be obtained by letting the sound velocity approach infinity.* The result will be the same as that obtained by assuming the fluid to be incompressible *ab initio*.

Cartesian coordinates x, y, and z will again be used, with z measured vertically upward. The velocity components in the directions of increasing x, y, and z will again be denoted by u, v, and w, all assumed to be small. The mean pressure $\bar{p}$, which is a function of z only, is related to the mean density $\bar{\rho}$, also a function of z only, by the hydrostatic condition

$$\bar{p}' = -g\bar{\rho}, \tag{4}$$

in which the prime indicates differentiation with respect to z. Since the velocity components are assumed to be small, the equation of continuity can be written in the linearized form

$$\frac{\partial \rho}{\partial t} + \bar{\rho}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + w\bar{\rho}' = 0, \tag{5}$$

* Under the very mild condition that $c_s'/c_s^3 \to 0$, in which c_s is the sound velocity, and c_s' is its gradient in the direction of the vertical.

in which ρ is now used for the density fluctuation, and is also assumed small. The equation of isentropy is

$$\frac{\partial p}{\partial t} + w\bar{p}' = c_s^2 \left(\frac{\partial \rho}{\partial t} + w\bar{\rho}'\right), \tag{6}$$

in which p is now the pressure fluctuation and c_s is the velocity of sound,* which is a function of z. In (5) and (6), products of the perturbation quantities u, v, w, ρ, and p with themselves or with their derivatives are neglected. They are therefore linearized equations, as are the other equations in this chapter. By virtue of (4), (6) can be written as

$$\frac{\partial p}{\partial t} - g\bar{\rho}w = c_s^2 \left(\frac{\partial \rho}{\partial t} + w\bar{\rho}'\right). \tag{7}$$

Finally, the linearized equations of motion are, with the terms representing the effects of viscosity neglected,

$$\bar{\rho}\,\frac{\partial}{\partial t}(u, v, w) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)p + (0, 0, -g\rho). \tag{8}$$

Cross differentiation of the first two equations contained in (8) produces

$$\frac{\partial}{\partial t}\left(\frac{\partial(\bar{\rho}u)}{\partial y} - \frac{\partial(\bar{\rho}v)}{\partial x}\right) = 0. \tag{9}$$

The dependent variables will be assumed to have an exponential time factor. Under that assumption, or if the motion is started from rest,

$$\frac{\partial(\bar{\rho}u)}{\partial y} - \frac{\partial(\bar{\rho}v)}{\partial x} = 0, \tag{10}$$

and a potential ϕ exists for u and v such that

$$\bar{\rho}u = \frac{\partial \phi}{\partial x}, \qquad \bar{\rho}v = \frac{\partial \phi}{\partial y}. \tag{11}$$

In any horizontal plane, the motion is irrotational as far as u and v are concerned, in the same way that Hele-Shaw† flows are irrotational with respect to velocity components in any plane parallel to the narrowly spaced plane boundaries. Equations (11) and the first two of Eqs. (8) produce

$$-\frac{\partial \phi}{\partial t} = p, \tag{12}$$

* The speed of sound is given by

$$c_s^2 = \left(\frac{d\bar{p}}{d\bar{\rho}}\right)_s,$$

the subscript indicating constant entropy. Hence (6) is the (linearized) statement of isentropic change of state.

† See Section 5, Chapter 5.

with the arbitrary function of integration $F(z, t)$ absorbed in $\partial\phi/\partial t$. With

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{13}$$

the equation of continuity can now be written in the form

$$-\nabla^2\phi = \bar{\rho}\frac{\partial w}{\partial z} + w\bar{\rho}' + \frac{\partial \rho}{\partial t}. \tag{14}$$

We have succeeded in expressing u, v, and p in terms of ϕ. Now between (7) and (14) the quantity $\partial\rho/\partial t$ can be eliminated. The result is, by virtue of (12),

$$\frac{\partial^2\phi}{\partial t^2} = -g\bar{\rho}w + c_s^2\left(\nabla^2\phi + \bar{\rho}\frac{\partial w}{\partial z}\right), \tag{15}$$

which relates ϕ to w. The third equation in (8) has so far not been used. By virtue of (12), that equation can be written as

$$-\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial z} - \bar{\rho}w\right) + g\rho = 0. \tag{16}$$

From (14) and (16) ρ can be eliminated to produce

$$\frac{\partial^3\phi}{\partial t^2\partial z} + g\nabla^2\phi + g\frac{\partial}{\partial z}(\bar{\rho}w) - \frac{\partial^2}{\partial t^2}(\bar{\rho}w) = 0. \tag{17}$$

Differentiation of (15) produces

$$\frac{\partial^3\phi}{\partial t^2\,\partial z} + g\frac{\partial}{\partial z}(\bar{\rho}w) - \frac{\partial}{\partial z}(c_s^2\,\nabla^2\phi) - \frac{\partial}{\partial z}\left(\bar{\rho}c_s^2\frac{\partial w}{\partial z}\right) = 0. \tag{18}$$

Presumably ϕ can be eliminated from (17) and (18) to give rise to a single equation in w. The result will be very involved. Since the perturbation quantities are to be assumed to have an exponential time factor and their dependence on x and y is to be specialized, the derivation of that single equation in w will be postponed until that dependence has been discussed. For the moment, we endeavour to find a convenient and more directly useful relation between ϕ and w than (17) and (18). Subtraction of (18) from (17) produces

$$-\nabla^2\left[g\phi + \frac{\partial}{\partial z}(c_s^2\phi)\right] + \frac{\partial^2}{\partial t^2}(\bar{\rho}w) - \frac{\partial}{\partial z}\left(\bar{\rho}c_s^2\frac{\partial w}{\partial z}\right) = 0,$$

and from (15) and (17) follows

$$-\frac{\partial^2}{\partial t^2}\left(g\phi + c_s^2\frac{\partial\phi}{\partial z}\right) - g^2\bar{\rho}w - c_s^2 g\bar{\rho}'w + c_s^2\frac{\partial^2}{\partial t^2}(\bar{\rho}w) = 0.$$

From the preceding two equations it results that

$$-(c_s^2)'\frac{\partial^2}{\partial t^2}\nabla^2\phi = \frac{\partial^4}{\partial t^4}(\bar{\rho}w) - \frac{\partial^2}{\partial t^2\,\partial z}\left(\bar{\rho}c_s^2\frac{\partial w}{\partial z}\right) + (g^2\bar{\rho} + c_s^2 g\bar{\rho}')\nabla^2 w - c_s^2\bar{\rho}\frac{\partial^2}{\partial t^2}\nabla^2 w. \tag{19}$$

The dependence of the perturbation quantities on x and y will now be discussed. In a large class of problems, the motion may be periodic in x and y, or a condition is to be satisfied along a closed curve in the x-y plane for all values of z, such as in the case of a cylindrical container. In this class of problems, (19) admits of a solution only if the variables x and y can be separated from z and t. Inspection of (12) through (19) reveals that a common factor $S(x, y)$ for all dependent variables will effect this separation if it satisfies the equation

$$(\nabla^2 + \alpha^2)S(x, y) = 0. \tag{20}$$

The quantity α^2 is the eigenvalue of (20) and the pertinent boundary conditions in any horizontal plane. It will be assumed, furthermore, that all the perturbation quantities possess the exponential time factor $e^{-i\sigma t}$, so that

$$(\phi, w) = e^{-i\sigma t}S(x, y)[\phi(z), w(z)]. \tag{21}$$

In fact, u, v, ρ, and p will then also have the factor $e^{-i\sigma t}\, S(x, y)$. With (20) available, ϕ can be eliminated from (15) and (19) to produce [Yih, 1960a]

$$\left(\frac{\alpha^2 c_s^2}{\sigma^2} - 1\right)[\sigma^2 c_s^2(\bar{\rho} w')' + (\sigma^4 - \alpha^2 g^2 - c_s^2\alpha^2\sigma^2)\bar{\rho} w - c_s^2\alpha^2 g\bar{\rho}' w] + (c_s^2)'\,\bar{\rho}(\alpha^2 g w - \sigma^2 w') = 0. \tag{22}$$

For three-dimensional sinusoidal waves, the appropriate form for $S(x, y)$ is $\exp i(kx + ly)$, in which k is the wave number in the x-direction and l that in the y-direction. If c is the phase velocity of the waves in the x-direction, then $\sigma = kc$, and*

$$(u, v, w, \rho, p) = [u(z), v(z), w(z), \rho(z), p(z)] \exp i(kx + ly - kct). \tag{23}$$

From now on, unless otherwise stated, u will be written for $u(z)$, etc., for the sake of brevity. From (11) and (12) it follows that

$$c\bar{\rho} u = p, \qquad lu = kv. \tag{24}$$

The third equation of (8) now assumes the form

$$ikc\bar{\rho} w = c(\bar{\rho} u)' + g\rho, \tag{25}$$

and the equation of isentropy the form

$$-ikcp - g\bar{\rho} w = c_s^2(-ikc\rho + w\bar{\rho}'). \tag{26}$$

Equation (15) now becomes

$$i\bar{\rho} u = \frac{kc_s^2\bar{\rho}}{(k^2 + l^2)c_s^2 - k^2c^2}\left(\frac{gw}{c_s^2} - w'\right). \tag{27}$$

* The time dependence of the form e^{-ikct} is used to emphasize the propagation of three-dimensional waves in the x-direction, in preparation for a kinematic relationship to be presented in Section 4, and in anticipation of the discussion of the stability of three-dimensional disturbances to be presented in Section 5 of Chapter 4. The reader can, if he so prefers, consider the time dependence to be of the form $e^{-i\sigma t}$, and read σ for the combination kc in what follows.

Equations (24), (25), (26), and (27) are recorded here for reference and general convenience.

With S having the exponential form, $\alpha^2 = k^2 + l^2$, and (22) becomes

$$\left[\frac{(k^2 + l^2)c_s^2}{k^2c^2} - 1\right]\{k^2c^2c_s^2(\bar{\rho}w')' + (kc)^4\bar{\rho}w - (k^2 + l^2)[(g^2 + k^2c^2c_s^2)\bar{\rho}w + c_s^2g\bar{\rho}'w]\} + (c_s^2)'\bar{\rho}[(k^2 + l^2)gw - k^2c^2w'] = 0. \quad (28)$$

If c_s is constant, this equation can be simplified to

$$(\bar{\rho}w')' - \frac{k^2 + l^2}{k^2}\left(k^2\bar{\rho} + \frac{g}{c^2}\bar{\rho}'\right)w + \frac{c^2}{c_s^2}\left(k^2 - \frac{(k^2 + l^2)g^2}{k^2c^4}\right)\bar{\rho}w = 0. \quad (29)$$

For an incompressible fluid, $c_s = \infty$, and we can assume that

$$gd^2(c_s^2)'c_s^{-4} \to 0 \qquad (d = \text{reference depth})$$

as $c_s \to \infty$. Then (28) can be written as

$$(\bar{\rho}w')' - \frac{k^2 + l^2}{k^2}\left(k^2\bar{\rho} + \frac{g}{c^2}\bar{\rho}'\right)w = 0, \quad (30)$$

which for two-dimensional flows ($l = 0$) has the well known form [Lamb, 1945, p. 378; Fjeldstad, 1933; Groen, 1948]

$$(\bar{\rho}w')' - \left(k^2\bar{\rho} + \frac{g}{c^2}\bar{\rho}'\right)w = 0. \quad (31)$$

For two-dimensional wave motion of a compressible fluid with constant c_s, (29) reduces to the form

$$(\bar{\rho}w')' - \left(k^2\bar{\rho} + \frac{g\bar{\rho}'}{c^2}\right)w + \frac{c^2}{c_s^2}\left(k^2 - \frac{g^2}{c^4}\right)\bar{\rho}w = 0. \quad (32)$$

This equation is also well known in the literature in various forms.

3. BOUNDARY CONDITIONS

If the fluid is obunded between two rigid barriers at $z = 0$ and $z = d$, the boundary conditions are

$$w(0) = 0, \qquad w(d) = 0. \quad (33)$$

If a rigid upper barrier does not exist, and the fluid is incompressible, the upper boundary condition can be obtained most simply by integrating (30) in the Stieltjes* sense across the free surface (for an infinitesimal interval

* If the integrand contains the derivative of a function which has a finite discontinuity (a step) at some place in the interval of integration, the integral in the ordinary sense does not exist. However, the integration of the derivative of a function should produce that function, if the definitions of derivative and of integration are properly generalized. Thus, the integration of the derivative of a function with steps should produce a function with the same steps. Such an integration is in essence one in the Stieltjes sense.

containing the point $z = d$). The result is, since $\bar{\rho} = 0$ above $z = d$,

$$-\bar{\rho}w'(d) + \frac{k^2 + l^2}{k^2c^2} g\bar{\rho}w(d) = 0, \tag{34}$$

or

$$w'(d) = \frac{k^2 + l^2}{k^2} \frac{g}{c^2} w(d). \tag{35}$$

Similarly, the condition at an interface separating two incompressible fluids is

$$(\bar{\rho}w')_u - (\bar{\rho}w')_l + \frac{k^2 + l^2}{k^2c^2} g(\bar{\rho}_l - \bar{\rho}_u)w = 0, \tag{36}$$

in which the subscripts indicate "upper" and "lower," respectively. This derivation is the simplest one, but may not be convincing to some readers. We shall therefore also give the conventional derivation. Since a free surface is an extreme case of an interface, we shall derive the interfacial condition, and obtain the free-surface condition as a special case. Let the displacement of the interface from its mean position be

$$\zeta = \zeta_0 \exp i(kx + ly - kct). \tag{37}$$

The kinematic condition at the interface is that the vertical velocity there must be equal to $\partial\zeta/\partial t$. Thus, at the interface,

$$w = -ikc\zeta_0. \tag{38}$$

On the other hand, the pressures in the two fluids must be the same at the interface. But the pressure consists of two parts: the perturbation pressure p and a hydrostatic part resulting from the displacement of the interface. Now, from the first of (24) and from (27),

$$p = -\frac{ickc_s^2\bar{\rho}}{(k^2 + l^2)c_s^2 - k^2c^2}\left(\frac{gw}{c_s^2} - w'\right). \tag{39}$$

Since $c_s = \infty$ for an incompressible fluid, and such a fluid is under consideration,

$$p = \frac{ick\bar{\rho}w'}{k^2 + l^2}. \tag{40}$$

The hydrostatic part of the total pressure is $-g\zeta_0\bar{\rho}$, which is equal to

$$-\frac{i\bar{\rho}}{kc} gw \tag{41}$$

by virtue of (36). Thus the interfacial condition is

$$\left(\frac{ck\bar{\rho}w'}{k^2 + l^2} - \frac{g\bar{\rho}w}{kc}\right)_u = \left(\frac{ck\bar{\rho}w'}{k^2 + l^2} - \frac{g\bar{\rho}w}{kc}\right)_l,$$

or

$$(\bar{\rho}w')_l - (\bar{\rho}w')_u = \frac{k^2 + l^2}{k^2c^2}(\bar{\rho}_l - \bar{\rho}_u)gw, \tag{42}$$

which is identical with (36). With $\bar{\rho}_u = 0$, (42) reduces to (35). Thus (35) and (36) have also been derived in the conventional manner. The fact that they can be derived from the differential equation by Stieltjes integration shows that they are *natural* boundary conditions.

In the case of a compressible fluid, it is unrealistic to talk about density discontinuities except for transient conditions. A surface of zero pressure can indeed exist at a finite height, but the density there is also zero, and the formulation of the free surface condition needs care. Let the atmosphere be finite in extent, so that $\bar{\rho}$ and $\bar{p}$ vanish at the height $z = d$. If we demand that the perturbation pressure p and the hydrostatic reduction in pressure due to the surface elevation should have a sum equal to zero, we see, from (39) and (41), that the condition is automatically satisfied, since $\bar{\rho} = 0$ at the free surface. We then have no upper boundary condition. Nevertheless the velocity component w must satisfy some condition. To find it, let $\bar{\rho}$ be small but non-zero at the free surface; then the condition just mentioned is

$$-\frac{i\bar{\rho}g}{kc}w - \frac{ickc_s^2\bar{\rho}}{(k^2 + l^2)c_s^2 - k^2c^2}\left(\frac{gw}{c_s^2} - w'\right) = 0, \tag{43}$$

which can be simplified to

$$w' = \frac{k^2 + l^2}{k^2}\frac{g}{c^2}w, \tag{44}$$

which is identical with (35). This is the upper boundary condition if $\bar{\rho}$ does not vanish at the free surface, and should be the upper boundary condition in the limiting case as $\bar{\rho} \to 0$ at the free surface. For two-dimensional waves, $l = 0$, and (44) becomes

$$w' = \frac{g}{c^2}w. \tag{45}$$

One objection to (44) as the appropriate upper boundary condition lies in the fact that at the free surface $\bar{\rho}$ and $\bar{p}$ are both zero, so that the perturbation quantities ρ and p are certainly not small compared with $\bar{\rho}$ and $\bar{p}$. This objection can be removed by the following argument.

For convenience, make the flow steady by superposing a uniform stream of velocity $-c$ on the wave motion. Then the Bernoulli equation

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + gz + \frac{(-c + u)^2 + v^2 + w^2}{2} = \text{constant} \tag{46}$$

is valid along any streamline, and in particular any streamline imbedded in the free surface. In (46), p is now the total pressure, ρ the total density and u, v, and w are the velocity components appearing on the left-hand side of (23). If the entropy along any streamline imbedded in the free surface is denoted by S, then along that streamline, from (1.5),

$$\frac{p}{\rho^\gamma} = \text{constant} \cdot e^{S/c_v}. \tag{47}$$

If S is finite, then since p is zero on the free surface, ρ must also be zero there, and consequently

$$\frac{p}{\rho} = 0, \tag{48}$$

since $\gamma > 1$ for all gases. Thus (46) becomes

$$gz + \frac{(-c + u)^2 + v^2 + w^2}{2} = \text{constant},$$

and with z replaced by ζ in (37) and with the quadratic terms in u^2, v^2, and w^2 neglected, the linearized condition is

$$g\zeta - cu = 0, \tag{49}$$

any constant other than zero on the right-hand side being unsuitable on account of the periodic nature of ζ and u. From (38) and (27), after the latter has been divided* throughout by $\bar{\rho}$, (49) assumes the form (43), which again can be simplified to the form (44). But now in the derivation use has not been made of the assumption that the pressure and density perturbations are small in comparison with $\bar{p}$ and $\bar{\rho}$.

So far we have shown that if the atmosphere is finite in extent and if the entropy is finite at its upper limit ($z = d$), (44) is the boundary condition at the free surface. It is known that a homentropic atmosphere (for which $\bar{S}$ is a finite constant) is finite in extent. Therefore the finiteness of d and that of $\bar{S}$ are not inconsistent. In fact, since, as is well known, an isothermal atmosphere is infinite in height, it is not consistent to assume a nonzero temperature at $z = d$ (finite) where $\bar{\rho}$ is assumed to be zero. If a nonzero temperature is assumed for the gas as $\bar{\rho} \to 0$, the thickness of the atmosphere must be infinite, and the entropy must approach infinity as $z \to \infty$, because $\bar{p}/\bar{\rho}$ is constant whereas

$$\bar{S} = c_v \ln (\bar{p}/\bar{\rho}^\gamma) + \text{finite constant}, \qquad \gamma > 1,$$

and $\bar{\rho} \to 0$ as $z \to \infty$. For an infinite atmosphere it is awkward to use (46), because both sides of it would be infinite. Indeed, it is awkward to think of a free surface at an infinite height. One device to remedy this situation is to impose (44) at some very large value of d, where $\bar{\rho}$ is not zero but very small. What actually happens above $z = d$ should have little effect on the flow below.

4. THREE-DIMENSIONAL WAVES

The most general form of the differential equation governing wave motion in a stratified fluid is (28), and the boundary conditions are either (33) or

$$w(0) = 0, \qquad w'(d) = \frac{k^2 + l^2}{k^2}\frac{g}{c^2} w(d). \tag{50}$$

* This division is valid so long as $\bar{\rho} \neq 0$. The result is valid even in the limit as $\bar{\rho} \to 0$.

In either case the study of three-dimensional waves can be reduced to that of two-dimensional waves, which are governed by

$$\left(\frac{c_s^2}{c^2} - 1\right)\{k^2c^2c_s^2(\bar{\rho}w')' + (kc)^4\bar{\rho}w - k^2[(g^2 + k^2c^2c_s^2)\bar{\rho}w + c_s^2g\bar{\rho}'w]\} + (c_s^2)'\bar{\rho}(k^2gw - k^2c^2w') = 0, \quad (51)$$

which is (28) with $l = 0$, and the boundary conditions (33) or

$$w(0) = 0, \qquad w' = \frac{g}{c^2}w. \quad (52)$$

With

$$k'^2 = k^2 + l^2 \qquad \text{and} \qquad k'c' = kc, \quad (53)$$

(28) reduces to

$$\left(\frac{c_s^2}{c'^2} - 1\right)\{(k'c')^2c_s^2(\bar{\rho}w')' + (k'c')^4\bar{\rho}w - k'^2[(g^2 + k'^2c'^2c_s^2)\bar{\rho}w + c_s^2g\bar{\rho}'w]\} + (c_s^2)'\bar{\rho}(k'^2gw - k'^2c'^2w') = 0, \quad (54)$$

in which the accents in k' and c' do not indicate differentiation. But with those accents dropped (54) is identical with (51). Furthermore, if the boundary conditions are (33), they are unchanged. If they are (50), they become

$$w(0) = 0, \qquad w'(d) = \frac{g}{c'^2}w(d), \quad (55)$$

or identical to (52) with the accent on c dropped. Thus, the differential system governing three-dimensional wave motion has been reduced to that governing two-dimensional wave motion. The development in this paragraph is after Squire [1933].

In fact, the reduction is quite obvious from the point of view of superposition of waves. With reference to Fig. 1, the two sets of lines forming the diamonds represent the ridges of the two sets of two-dimensional waves with the same wavelength λ', which together form the three-dimensional wave pattern. The wave number k' of the two-dimensional waves is related to λ' through

$$\lambda' = \frac{2\pi}{k'}. \quad (56)$$

If λ_k and λ_l are the wave lengths in the x- and y-directions, respectively, then

$$\lambda_k = \frac{2\pi}{k}, \qquad \lambda_l = \frac{2\pi}{l}. \quad (57)$$

It is evident that

$$\sin\theta = \frac{\lambda_l}{\sqrt{\lambda_k^2 + \lambda_l^2}}, \quad (58)$$

which, by virtue of (57), can be written as

$$\sin \theta = \frac{k}{\sqrt{k^2 + l^2}}. \tag{59}$$

Now

$$\lambda' = \lambda_k \sin \theta. \tag{60}$$

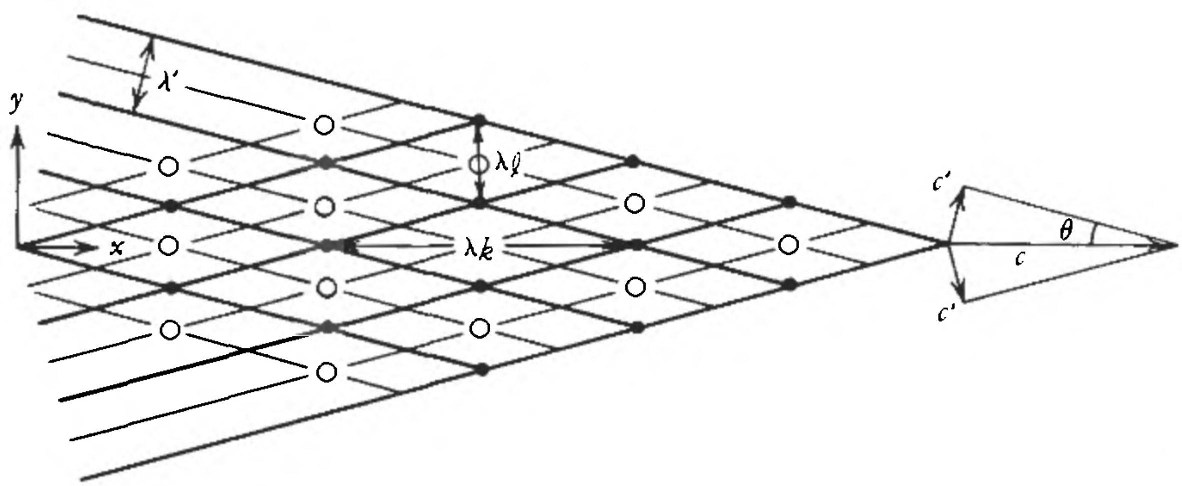
With (57) and (59), this becomes

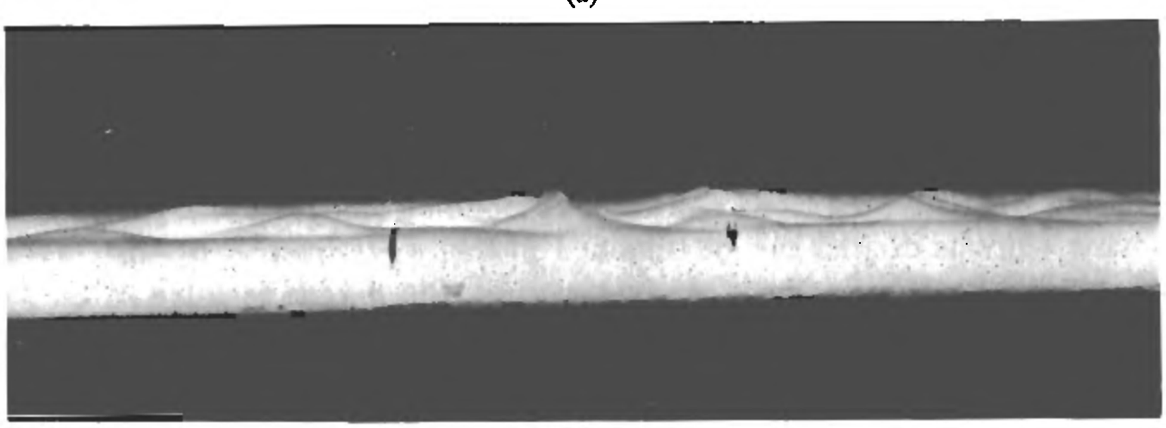
$$\frac{k'k}{\sqrt{k^2 + l^2}} = k,$$

or

$$k' = \sqrt{k^2 + l^2},$$

which is the first of (53). The second of (53) follows from the fact that the velocity of propagation c of the three-dimensional waves must have its component in the direction of propagation of either set of the two-dimensional





(b)

FIGURE 1. (a) Sketch showing a three-dimensional wave pattern and the kinematic relationship between its phase velocity and that of a two-dimensional component wave. Dots indicate high values and circles low values of the displacement ζ. (b) A three-dimensional wave pattern in a stream of clay-laden water flowing under clear water. *(Photograph courtesy of Professors A. T. Ippen and D. F. Harleman.)*

In either case the study of three-dimensional waves can be reduced to that of two-dimensional waves, which are governed by

$$\left(\frac{c_s^2}{c^2} - 1\right)\{k^2c^2c_s^2(\bar{\rho}w')' + (kc)^4\bar{\rho}w - k^2[(g^2 + k^2c^2c_s^2)\bar{\rho}w + c_s^2g\bar{\rho}'w]\} + (c_s^2)'\bar{\rho}(k^2gw - k^2c^2w') = 0, \quad (51)$$

which is (28) with $l = 0$, and the boundary conditions (33) or

$$w(0) = 0, \qquad w' = \frac{g}{c^2}w. \quad (52)$$

With

$$k'^2 = k^2 + l^2 \qquad \text{and} \qquad k'c' = kc, \quad (53)$$

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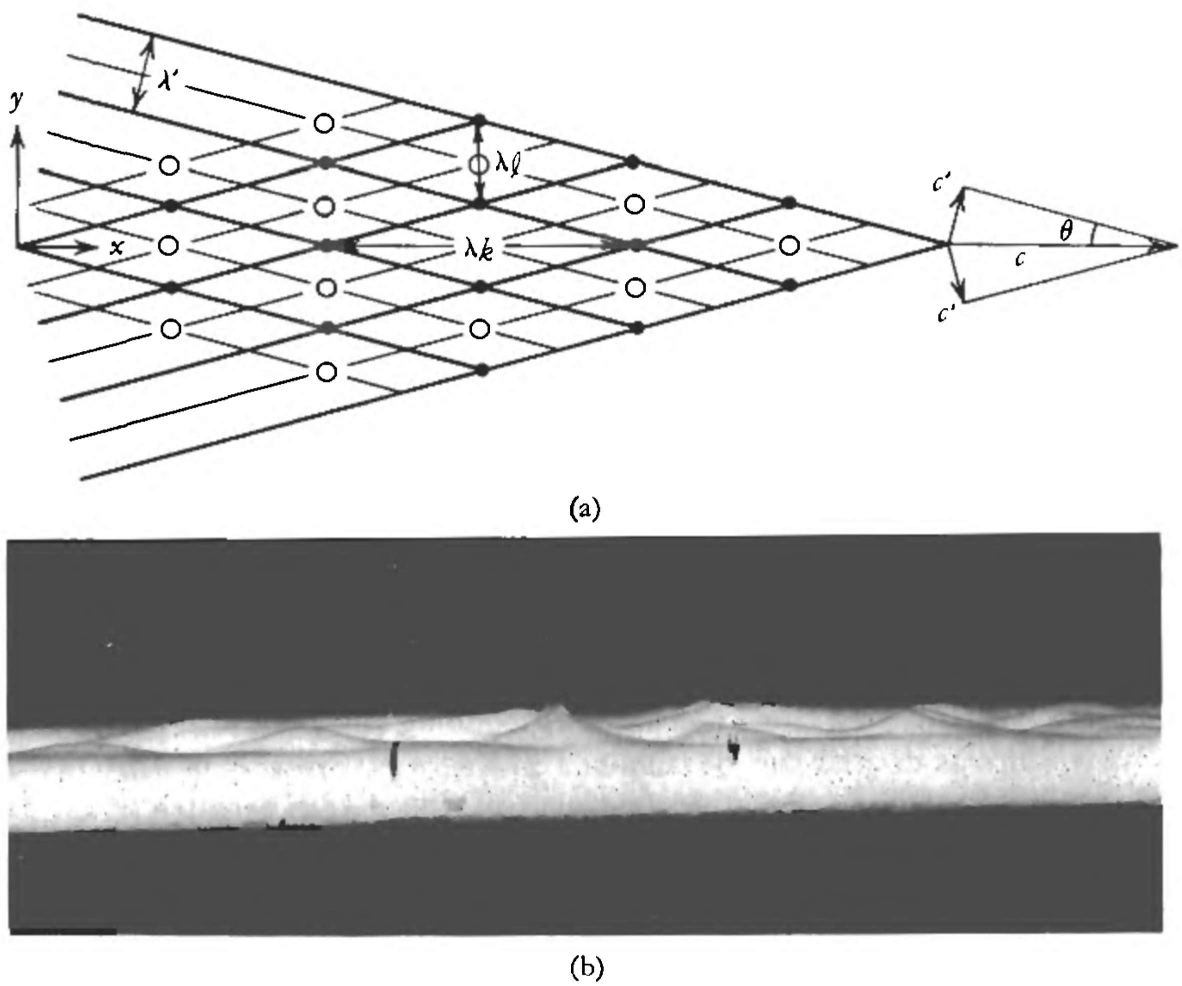

FIGURE 1. (a) Sketch showing a three-dimensional wave pattern and the kinematic relationship between its phase velocity and that of a two-dimensional component wave. Dots indicate high values and circles low values of the displacement ζ. (b) A three-dimensional wave pattern in a stream of clay-laden water flowing under clear water. *(Photograph courtesy of Professors A. T. Ippen and D. F. Harleman.)*

waves exactly equal to c'. This leads to

$$c \sin \theta = c', \tag{61}$$

or

$$ck = c'k'.$$

Thus, unless the boundary topography demands otherwise, the study of three-dimensional waves can be reduced to that of two-dimensional waves, within the framework of a linear theory.

5. EFFECT OF COMPRESSIBILITY IN AN ISOTHERMAL ATMOSPHERE

The effect of compressibility can be assessed in a simple manner if and only if the sound speed remains constant from elevation to elevation. The results of Section 4 enable us to consider only two-dimensional waves without loss of generality.

The equations to be compared in an assessment of the effect of compressibility are (31) and (32). Let a fictitious wave number k_i be defined by

$$k_i^2 = k^2 + \frac{c^2}{c_s^2}\left(\frac{g^2}{c^4} - k^2\right). \tag{62}$$

Then (32) becomes

$$(\bar{\rho}w')' - \left(k_i^2\bar{\rho} + \frac{g}{c^2}\bar{\rho}'\right)w = 0. \tag{63}$$

Thus k_i is the wave number of a fictitious wave motion of an incompressible fluid, which corresponds to the wave motion of the compressible fluid under examination, and the study of wave motion in an isothermal atmosphere (of a compressible fluid) has been reduced to the study of wave motion of an incompressible fluid.

The finding of c for a given k in (32) can then be accomplished by the following procedure. Assume k_i, find c as for an incompressible fluid, and go back to (62) and find k. So long as the assumption of k_i is such that the resulting k is real, we get one pair of corresponding values for c and k, and as many pairs as desired can be obtained in this way. Since k_i is fictitious, it does not have to be real (k_i^2 can be negative). But if k_i^2 is negative k^2 can be positive only if

$$c^2 > c_s^2.$$

In fact, as will be shown in Section 10, g/k is the c^2 for free-surface waves in a semi-infinite fluid extending from $z = 0$ to $z = -\infty$, and the c^2 for internal or interfacial waves of the gravity type is usually smaller, so that $k_i^2 > k^2$ in most cases of practical interest. It will be shown in the next section that c decreases with the wave number for an incompressible fluid. Therefore, whenever $k_i^2 > k^2$, the phase velocity is smaller for a compressible fluid (with constant c_s) than for an incompressible fluid. (This results from a comparison of (31) with (63).) Thus, whenever c_s is constant and $k_i^2 > k^2$, the effect of compressibility is to *reduce* the phase velocity.

6. THE EIGENVALUE PROBLEM FOR THE GENERAL CASE

Now that it has been shown that the study of three-dimensional wave motion can be reduced to that of two-dimensional wave motion, attention will be focused on the latter throughout the rest of the chapter. The general differential equation governing two-dimensional wave motion is (51), which, after division by $k^2c_s^2(c_s^2 - c^2)$, assumes the form

$$(\bar\rho w')' + \left\{\left[k^2\left(\frac{c^2}{c_s^2} - 1\right) - \frac{g^2}{c^2c_s^2} + \frac{(c_s^2)'g}{c_s^2(c_s^2 - c^2)}\right]\bar\rho - \frac{g\bar\rho'}{c^2}\right\}w - \frac{(c_s^2)'c^2}{c_s^2(c_s^2 - c^2)}\bar\rho w' = 0.$$

After multiplication by $c_s^2/(c_s^2 - c^2)$, this equation becomes

$$\left(\frac{c_s^2}{c_s^2 - c^2}\bar\rho w'\right)' - \left\{\left[k^2 - \frac{(c_s^2)'g}{(c_s^2 - c^2)^2}\right]\bar\rho + \frac{g(g\bar\rho + c_s^2\bar\rho')}{c^2(c_s^2 - c^2)}\right\}w = 0. \tag{64}$$

This equation involves the derivative of the square of the speed of sound with respect to z, or, equivalently, the temperature gradient. It will be desirable to express $(c_s^2)'$ in terms of $\bar\rho'$. This is possible for ideal gases because their equation of state is known and, together with the equation of hydrostatics, can be used to relate the temperature to the density. Thus, if γ is assumed constant,

$$c_s^2 = \gamma R\bar T = \gamma\frac{\bar p}{\bar\rho},$$

and

$$(c_s^2)' = \gamma\left(\frac{\bar p'}{\bar\rho} - \frac{\bar p\bar\rho'}{\bar\rho^2}\right).$$

But according to (4), $\bar p' = -g\bar\rho$. Hence

$$(c_s^2)' = -\gamma\left(g + \frac{\bar p\bar\rho'}{\bar\rho^2}\right) = -\gamma g - \frac{c_s^2\bar\rho'}{\bar\rho}. \tag{65}$$

Substituting (65) into (64), we have, for ideal gases,

$$\left(\frac{c_s^2}{c_s^2 - c^2}\bar\rho w'\right)' - \left[k^2\bar\rho + \frac{c_s^2 g(g\bar\rho + c_s^2\bar\rho')}{c^2(c_s^2 - c^2)^2} + \frac{(\gamma - 1)g^2\bar\rho}{(c_s^2 - c^2)^2}\right]w = 0. \tag{66}$$

The quantity $g\bar\rho + c_s^2\bar\rho'$, which occurs both in (64) and (66), is always negative for a stable atmosphere (with entropy increasing upward) because it is equal to $-\bar{p}' + c_s^2\bar{\rho}'$, $\bar{\rho}'$ is negative, and $c_s^2 = (dp/d\rho)_S$ is greater than $\bar{p}'/\bar{\rho}' = d\bar{p}/d\bar{\rho}$ in the atmosphere.

The boundary conditions are

$$w(0) = 0 \quad \text{and} \quad w(d) = 0 \tag{67}$$

for rigid boundaries, and

$$w(0) = 0 \quad \text{and} \quad w'(d) = \frac{g}{c^2}\, w(d) \tag{68}$$

for a rigid lower boundary and a free surface. Equation (64) or (66), together with (67) or (68), constitutes a differential system which defines an eigenvalue problem.

For a given value of k and a given stratification (so that $\bar{\rho}$ and c_s^2 are given functions of z), the governing differential system demands that c assume certain values (which are then the eigenvalues of c for the differential system), if w is not to be identically zero.

The mean pressure, mean density, and mean entropy (S) are related by

$$\frac{\bar{p}}{\bar{\rho}^{\gamma}} = \frac{\bar{p}_0}{\bar{\rho}_0^{\gamma}}\, e^{S/c_v},$$

in which the subscript 0 indicates that the quantity involved is taken at $z = 0$, and S is taken to be zero at $z = 0$. Since $d\bar{p} = -g\bar{\rho}\, dz$, the quantity $g\bar{\rho} + c_s^2\bar{\rho}'$ is equal to $-c_s^2\bar{\rho}S'/c_p$, where $S' = dS/dz$, and hence is always negative for a stable atmosphere (for which S' is positive). Examples will be given to show that the eigenvalues of c^2 may be greater or smaller than c_s^2. In fact, if the boundary conditions are given by (67), with the Sturm-Liouville theory in mind one can readily see that these eigenvalues must have the limit points zero and infinity. For those values of c^2 very much smaller than c_s^2 and for $gd \ll c_s^2$, (66) can be written simply as

$$(\bar{\rho}w')' - \left(k^2 - \frac{gS'}{c^2 c_p}\right)\bar{\rho}w = 0. \tag{66a}$$

If $\bar{\rho}_p$ is the potential density of the fluid—that is, the density of any part of the fluid when it is brought isentropically to the level $z = 0$—then

$$\frac{\bar{p}_0}{\bar{\rho}_p^{\gamma}} = \frac{\bar{p}}{\bar{\rho}^{\gamma}} = \frac{\bar{p}_0}{\bar{\rho}_0^{\gamma}}\, e^{S/c_v},$$

and

$$\bar{\rho}_p = \bar{\rho}_0\, e^{-S/c_p}.$$

Thus

$$\bar{\rho}_p' = -\bar{\rho}_p \frac{S'}{c_p},$$

and the differential equation above for small c^2 can be written as

$$(\bar{\rho}w')' - \left(k^2\bar{\rho} - \frac{g\bar{\rho}_p'}{c^2}\frac{\bar{\rho}}{\bar{\rho}_p}\right)w = 0. \tag{66b}$$

Note that for an incompressible fluid the differential equation is (31), to which the one just given for compressible fluids can be reduced by equating $\bar{\rho}$ to $\bar{\rho}_p$.

If c^2 is very much greater than c_s^2, gd, or $c_s^2\bar{\rho}'/\bar{\rho}$, (66) assumes the form

$$(c_s^2\bar{\rho}w')' + k^2c^2\bar{\rho}w = 0. \tag{66c}$$

7. THE EIGENVALUE PROBLEM FOR WAVE MOTION IN AN INCOMPRESSIBLE FLUID

In the case of a continuously stratified incompressible fluid, the governing differential system consists of (31) and (67) or (31) and (68):

$$(\bar{\rho}w')' - \left(k^2\bar{\rho} + \frac{g}{c^2}\,\bar{\rho}'\right)w = 0,$$

$$w(0) = 0, \qquad \text{and} \qquad w(d) = 0 \qquad \text{or} \qquad w'(d) = \frac{g}{c^2}\,w(d).$$

This system will be called System I for convenience. To study the existence and distribution of the eigenvalues for c^2 and the character of the eigenfunctions, the oscillation theorem of Sturm [see Bôcher, 1917, pp. 66–67] is very useful. This theorem will now be presented.

Consider the system

$$\frac{d}{dz}(Kf') - Gf = 0, \tag{69}$$

$$\alpha' f(a) - \alpha f'(a) = 0, \tag{70}$$

$$\beta' f(b) + \beta f'(b) = 0, \tag{71}$$

in which K and G are functions of z and the parameter λ, which do not increase as λ increases from Λ_1 to Λ_2, K being always positive, and α, α', β, and β' are functions of λ. If $\beta = 0$, or β does not vanish and $K(b)\beta'/\beta$ is a decreasing function of λ, and if, in addition,

$$\lim_{\lambda\to\Lambda_2}\left(\frac{-\max G}{\max K}\right) = +\infty, \tag{72}$$

$$\lim_{\lambda\to\Lambda_1}\left(\frac{-\min G}{\min K}\right) = -\infty, \tag{73}$$

the system has an infinite number of eigenvalues $\lambda_0, \lambda_1, \ldots,$ between Λ_1 and Λ_2, in ascending order of magnitude. Each of the eigenfunctions f_0, $f_1, \ldots,$ which are solutions of the system for $\lambda = \lambda_0, \lambda_1, \ldots,$ has a number of zeros between a and b (with $z = a$ and $z = b$ excluded) exactly equal to its respective index.

Since in our discussion $w(0) = 0$, we shall, whenever the above theorem is applicable, designate the eigenfunctions of the differential system governing wave motion by w_1, w_2, etc., for the eigenvalues λ_1, λ_2, etc. (of c^{-2}), each of which then has a number of zeros in the interval $0 \leq z < d$ exactly equal to its respective index. Thus this index, denoted by n, indicates the number of nodal planes in the fluid in the interval $0 \leq z < d$, and therefore the mode of the pertaining wave motion. For a given wave number k, there are infinitely many modes if there are infinitely many eigenvalues for c^2.

For System I, the existence of infinitely many eigenvalues for c^2 for each value of k follows directly from Sturm's oscillation theorem. The differential equation, (31), is certainly of the type (69). Being equal to $\bar{\rho}$, K is always positive and does not increase as λ, identified with c^{-2}, increases. G, being equal to

$$k^2\bar{\rho} + \frac{g}{c^2}\bar{\rho}' = k^2\bar{\rho} + g\lambda\bar{\rho}',$$

and $\bar{\rho}'$ being negative, decreases as λ increases, as required. Since $a = 0$ and $b = d$, the boundary conditions are such that either

$$\alpha = 0, \qquad \alpha' = 1, \qquad \beta = 0, \qquad \beta' = 1,$$

or

$$\alpha = 0, \qquad \alpha' = 1, \qquad \beta = 1, \qquad \beta' = -\frac{g}{c^2} = -g\lambda.$$

Thus either $\beta = 0$ or

$$\frac{K(b)\,\beta'}{\beta} = -g\lambda\,\bar{\rho}(d),$$

which is a decreasing function of λ (as λ increases), as required. Furthermore, it can be seen easily that $\Lambda_1 = -\infty$, and $\Lambda_2 = \infty$. Thus there are infinitely many eigenvalues between $-\infty$ and ∞ for c^{-2}. Thus c^2 lies between $-\infty$ and ∞ also. As λ increases, hence c^2 decreases, more and more zeros will appear in $0 \leq z < d$, and hence more and more nodal planes in the fluid.

The reality of c^2 can be simply demonstrated. If the differential equation of System I is multiplied by w^*, the complex conjugate of w, and integrated (by parts for the first term) between 0 and d, the result is

$$\bar{\rho}w^*w'|_0^d - \int_0^d \bar{\rho}|w'|^2\,dz - k^2\int_0^d \bar{\rho}|w|^2\,dz - \frac{g}{c^2}\int_0^d \bar{\rho}'|w|^2\,dz = 0.$$

The first term is zero if $w(d) = 0$, and is

$$\frac{g\bar{\rho}}{c^2}\,|w(d)|^2.$$

In either case c^2 is real. Furthermore, if $\bar{\rho}'$ is negative throughout, c^2 is positive, and if $\bar{\rho}'$ is positive throughout and the upper boundary is rigid, c^2 is negative. If the upper boundary is rigid and $\bar{\rho}'$ is partly positive and partly negative, for each value of k and a given number of zeros of w in the interval $0 \leq z < d$, there are two eigenvalues for c^2, one positive and one negative.

For a mathematical proof of this statement, see Bôcher [1917, pp. 70–72]. If $\bar{\rho}$ remains the same but the direction of gravitational acceleration is reversed, the sign of c^2 will be changed. In particular, the rate of amplification $k|c|$, in which c is imaginary, for an entirely unstable configuration is exactly equal to kc_1, where c_1 is the phase velocity (taken as positive in this connection) of the reversed, hence stable, configuration.

If $w(d) = 0$, the nth mode is the mode for which w has n zeros in $0 \leq z < d$, or $n + 1$ zeros in $0 \leq z \leq d$. If $w'(d) = (g/c^2)w(d)$, then the nth mode is characterized by n zeros of w in $0 \leq z \leq d$. This convention will be adopted for System I, and any system which is strictly a Sturm system as described earlier in this section. Since $\Lambda_2 = \infty$ corresponds to $c^2 = 0$, zero is a limit point of the eigenvalues for c^2, in the neighborhood of which there are infinitely many eigenvalues for c^2. Sturm's theorem shows that as $c^2 \to 0$, the number of zeros for w increases indefinitely, so that for smaller and smaller phase velocities the number of nodal planes increases more and more.

8. DEPENDENCE OF PHASE VELOCITY ON WAVELENGTH FOR SYSTEM I

The development in Section 4 enables us to concentrate on two-dimensional waves. For a continuously stratified incompressible fluid confined between two rigid boundaries, it has been shown by Groen [1958] and independently by Yih [1960a] that the phase velocity c increases with wavelength. The dependence of c on wavelength for waves propagating in an incompressible fluid with a free surface or in a compressible fluid has not been discussed in the literature. (See, however, the footnote in Section 12.3.) The discussion in this section is confined to the case of an incompressible fluid.

The comparison theorems of Sturm are indispensable for the present discussion. Consider two systems [see Bôcher, 1917, pp. 58–63],

$$(K_1 f')' - G_1 f = 0, \tag{74}$$

$$f(a) = \alpha_1, \qquad f'(a) = \alpha_1', \tag{75}$$

and

$$(K_2 f')' - G_2 f = 0, \tag{76}$$

$$f(a) = \alpha_2, \qquad f'(a) = \alpha_2', \tag{77}$$

in which K_1, K_2, G_1, and G_2 are functions of z, the accents on the dependent variable f indicate differentiation with respect to z, and

$$K_1 \geq K_2 > 0, \qquad G_1 \geq G_2. \tag{78}$$

It is assumed that

$$|\alpha_1| + |\alpha_1'| \neq 0, \qquad |\alpha_2| + |\alpha_2'| \neq 0, \tag{79}$$

that the equalities $K_1 = K_2$ and $G_1 = G_2$ do not hold in any part of the interval ab, and that $G_1 = G_2 = 0$ does not hold in any part of the interval. Furthermore, if $\alpha_1 \neq 0$, it is assumed that $\alpha_2 \neq 0$ and

$$\frac{K_1(a)\,\alpha_1'}{\alpha_1} \geq \frac{K_2(a)\,\alpha_2'}{\alpha_2}. \tag{80}$$

(If $\alpha_1 = 0$, no supplementary assumption need be made.)

Under the preceding assumptions, Sturm's comparison theorems hold. The first theorem can be stated as follows: If the solution f_1 of the first system has a certain number of zeros distinct from a in the interval ab ($a < z \leq b$), the solution f_2 of the second system must have at least as many zeros in this interval, and if z_1, z_2, z_3, etc. are the zeros of f_1 in ascending order, and z_1', z_2', z_3', etc. are those of f_2,

$$z_i' < z_i \tag{81}$$

for all values of i corresponding to a zero of f_1 and f_2.

Sturm's second theorem states: If, in addition to the preceding hypotheses, it is assumed that $f_1(b) \neq 0$ and $f_2(b) \neq 0$, then

$$\frac{K_1(b)\,f_1'(b)}{f_1(b)} > \frac{K_2(b)\,f_2'(b)}{f_2(b)}, \tag{82}$$

provided that the solutions f_1 and f_2 of the two systems have the same number of zeros in the interval $a < z \leq b$.

Sturm's comparison theorems can be applied directly to find how the phase velocity of gravity waves propagating in an incompressible fluid varies with their wavelength. If the fluid is confined between two rigid boundaries, the eigenfunction w must have one zero at $z = 0$, and another at $z = d$. With $\bar{\rho}$ a given function of z, for a greater wave number k (hence a smaller wavelength) the phase velocity* c is always smaller, provided that the number $(n + 1)$ of zeros of w (or the number of nodal planes) in the interval $0 \leq z \leq d$ remains the same. Comparison of System I with (74) and (75) reveals that $K = \bar{\rho}$, and hence is the same for all wavelengths, and that

$$G(z) = k^2\bar{\rho} + \frac{g}{c^2}\,\bar{\rho}',$$

$$a = 0, \qquad b = d.$$

If for $k = k_2$ the eigenvalue for c for the nth mode (with $n + 1$ zeros in $0 \leq z \leq d$) is c_{2n}, then for $k = k_1 > k_2$, the corresponding eigenvalue c_{1n}

* Or, rather, the square of the phase velocity, since c may be positive or negative, and it is c^2 that is really relevant. For convenience, c will always be considered positive.

must be less than c_{2n}, for otherwise G_1 would be uniformly greater than G_2, and (81) would give

$$z_{n+1} > z'_{n+1} = d,$$

contrary to hypothesis.

If the upper boundary condition is

$$w'(d) = \frac{g}{c^2} w(d),$$

the same result follows, for (82) now gives

$$\frac{\bar{\rho}(d)\, g}{c_{1n}^2} > \frac{\bar{\rho}(d)\, g}{c_{2n}^2}, \qquad \text{or} \qquad c_1 < c_2,$$

provided the number n of zeros of w in $0 \leq z \leq d$ (or the number $n - 1$ of zeros in $0 < z \leq d$) remains the same. I am not aware of any previous application of Sturm's second theorem to obtain the result that c decreases as k increases in wave motion of a heterogeneous liquid with a free surface.

The results so far obtained in this section are also valid for a compressible fluid, provided c^2 and gd are both much smaller than c_s^2 because the governing differential equation, (66a) or (66b), is similar to (31) in form, and all the arguments advanced in this section can be carried over. The same is true if c^2 is very large, since (66c) is of such a form that evidently the same conclusions are obtained.

Sturm's comparison theorems can also be used to show the effect of a change in the density distribution on the eigenvalues of c^2. If k^2 is kept constant and $\bar{\rho}$ is increased everywhere by a constant, $\bar{\rho}'$ is unchanged and K is increased. Then c^2 for a definite mode must decrease. For, if c^2 should increase, G would definitely increase, and it would follow from Sturm's first comparison theorem that the condition $w(d) = 0$ could not be satisfied in the case of a rigid upper boundary, or from his second theorem that c^2 would decrease in the case of a free surface at $z = d$, leading to a contradiction. The same is true if $\bar{\rho}$ is increased everywhere and $|\bar{\rho}'|$ is decreased everywhere. On the other hand, if $\bar{\rho}$ is decreased and $|\bar{\rho}'|$ increased everywhere, c^2 will increase. All these are quite understandable from the physical point of view, for $\bar{\rho}$ represents the inertia of the fluid and $g\bar{\rho}'$ (with $\bar{\rho}'$ negative) the restoring force responsible for the existence of wave motion, and the frequency of the motion will increase with smaller inertia and greater restoring force, and decrease otherwise. Since k is kept constant, an increase or decrease in the frequency means the same in $|c|$.

9. WAVE MOTION IN A BOUNDED ISOTHERMAL ATMOSPHERE

The differential equation governing wave motion in an isothermal fluid is (32). The governing differential system is, for an atmosphere bounded at $z = 0$ and $z = d$,

$$(\bar{\rho}w')' - \left(k_i^2\bar{\rho} + \frac{g}{c^2}\rho'\right)w = 0,$$

$$w(0) = 0, \quad \text{and} \quad w(d) = 0,$$

$$k_i^2 = k^2 + \frac{c^2}{c_s^2}\left(\frac{g^2}{c^4} - k^2\right).$$

This system will be called System II. For each value of k_i^2, there are infinitely many eigenvalues for c^2, corresponding to infinitely many modes of wave motion, with an increasing number of nodal planes as $c^2 \to \infty$ or $c^2 \to 0$. But of course the real question is: For a given value of k (not k_i), how are the eigenvalues distributed, and what happens to the characterization of the eigenfunctions by the numbers of their zeros in $0 \leq z \leq d$? This question was discussed by Eckart [1960, pp. 138–143], but only incompletely, because terms containing the factor g/c_s^2 (his Γ) were omitted by him.

For an isothermal atmosphere of a gas, the equation of state

$$\frac{\bar{p}}{\bar{\rho}} = R\bar{T} \qquad (\bar{T} = \text{constant}) \tag{83}$$

and the equation of hydrostatics—that is, Eq. (4)—give

$$R\bar{T}\,d\bar{\rho} = -g\bar{\rho}\,dz, \tag{84}$$

integration of which produces

$$\bar{\rho} = \bar{\rho}_0\,e^{-gz/R\bar{T}} = \bar{\rho}_0\,e^{-\beta z}, \tag{85}$$

in which β is written for $g/R\bar{T}$ for simplicity. The differential equation in System II can then be written in the form

$$w'' - \beta w' - \left(k_i^2 - \frac{\beta g}{c^2}\right)w = 0, \tag{86}$$

the general solution of which is

$$w = Ae^{\lambda_1 z} + Be^{\lambda_2 z}, \tag{87}$$

in which λ_1 and λ_2 are roots of the indicial equation

$$\lambda^2 - \beta\lambda + \frac{g\beta}{c^2} - k_i^2 = 0, \tag{88}$$

that is,

$$(\lambda_1, \lambda_2) = \frac{1}{2}\left[\beta \pm \left(\beta^2 + 4k_i^2 - \frac{4\beta g}{c^2}\right)^{1/2}\right]. \tag{89}$$

Since the boundary conditions are $w(0) = w(d) = 0$,

$$B = -A, \tag{90}$$

and

$$\beta^2 + 4k_i^2 - \frac{4\beta g}{c^2} = -\frac{4n^2\pi^2}{d^2}, \tag{91}$$

or, in terms of k^2,

$$\frac{4k^2c^4}{c_s^2} - \left(\beta^2 + \frac{4n^2\pi^2}{d^2} + 4k^2\right)c^2 + 4\left(\beta g - \frac{g^2}{c_s^2}\right) = 0. \tag{92}$$

The integral number n is the number of zeros of w in $0 \leq z < d$. That is to say, $n + 1$ is the number of zeros in the closed interval $0 \leq z \leq d$. The solutions of (92) are

$$c^2 = (8k^2)^{-1}c_s^2\{\beta'^2 \pm [\beta'^4 - 64k^2c_s^{-2}g(\beta - gc_s^{-2})]^{1/2}\}, \tag{93}$$

with

$$\beta'^2 = \beta^2 + \frac{4n^2\pi^2}{d^2} + 4k^2. \tag{94}$$

Now since $\beta = g/R\bar{T} = \gamma g/c_s^2$, the quantity $\beta - gc_s^{-2}$ is positive. In order to prove that c^2 is real and positive, we have to prove that

$$\left(\beta^2 + \frac{4n^2\pi^2}{d^2} + 4k^2\right)^2 > \frac{64k^2g}{c_s^2}\left(\beta - \frac{g}{c_s^2}\right). \tag{95}$$

Since $\beta = \gamma g c_s^{-2}$, the inequality to be proved is

$$\left(\beta^2 + \frac{4n^2\pi^2}{d^2} + 4k^2\right)^2 > 64\left(\frac{\gamma - 1}{\gamma^2}\right)k^2\beta^2.$$

Since

$$\gamma = \frac{N + 2}{N},$$

in which N is the degree of freedom of the gas molecules,

$$\frac{\gamma - 1}{\gamma^2} = \frac{2N}{(N + 2)^2} = \frac{6}{25}, \quad \frac{10}{49}, \quad \frac{12}{64} < \frac{1}{4}$$

for $N = 3$, 5, and 6, corresponding to monatomic, diatomic, and multiatomic molecules, respectively. Thus the inequality to be proved can be replaced by

$$\left(\beta^2 + \frac{4n^2\pi^2}{d^2} + 4k^2\right)^2 > 16k^2\beta^2,$$

which is obviously true since

$$(\beta^2 + 4k^2)^2 \geq 16k^2\beta^2.$$

Thus there are two real and positive values for c^2 for each (real and positive) value of k^2, and for each value of n (that is, for each assigned number of nodal planes in $0 \leq z < d$). When $k^2 \to \infty$, (92) indicates that $c^2 \to 0$ or c_s^2. When $k^2 \to 0$, (92) shows that $c^2 \to c_1^2 < c_s^2$ for one branch, and $c^2 \to \infty$ for the other. Thus the graph of k versus c is of the general trend indicated in Fig. 2. Further proof of the trend follows.

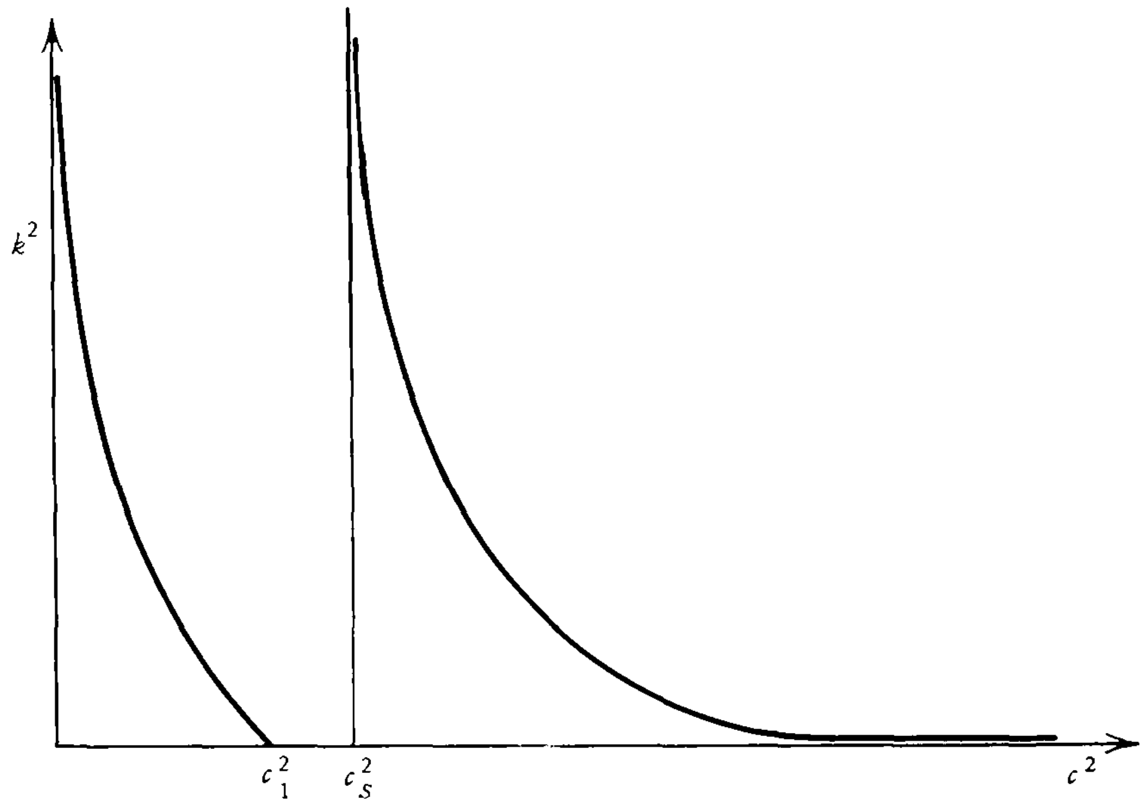

FIGURE 2. Variation of c^2 with k^2 for wave motion in an isothermal atmosphere between two rigid horizontal boundaries.

If the two roots of (93) are denoted by c_2^2 and c_1^2, with the positive sign in (93) used for c_2^2 and the negative sign for c_1^2, then $c_2^2 > c_1^2$, and

$$c_1^2 + c_2^2 = \frac{c_s^2\beta'^2}{4k^2}, \qquad c_1^2c_2^2 = \frac{c_s^2}{k^2}\left(\beta g - \frac{g^2}{c_s^2}\right) = \frac{(\gamma - 1)g^2}{k^2}. \tag{96}$$

Since

$$c_2^2 > \frac{c_s^2\beta'^2}{8k^2} > \frac{c_s^2}{2}\left(1 + \frac{\beta^2}{4k^2}\right) = \frac{c_s^2}{2} + \frac{\gamma^2 g^2}{8k^2c_s^2}, \tag{97}$$

it follows from the second equation in (96) that

$$c_2^2 > \frac{c_s^2}{2} + \frac{\gamma^2}{8(\gamma - 1)}\frac{c_1^2c_2^2}{c_s^2}. \tag{98}$$

Now it has been shown that

$$\frac{\gamma^2}{4(\gamma - 1)} > 1. \tag{99}$$

From (96), (98), and (99) it will be shown that $c_2^2 > c_s^2 > c_1^2$. It follows from the first of (96) that

$$c_1^2 + c_2^2 > c_s^2 + \frac{\gamma^2 g^2}{4c_s^2 k^2} = c_s^2 + \frac{\gamma^2}{4(\gamma - 1)c_s^2} c_1^2 c_2^2.$$

Thus

$$c_2^2\left(1 - \frac{\gamma^2 c_1^2}{4(\gamma - 1)c_s^2}\right) > c_s^2 - c_1^2 > 0,$$

or

$$\frac{c_2^2}{c_s^2}\left(1 - \frac{\gamma^2 c_1^2}{4(\gamma - 1)c_s^2}\right) > 1 - \frac{c_1^2}{c_s^2}. \tag{100}$$

It then follows from (99) that $c_1^2 < c_s^2$, for otherwise $c_2^2 > c_1^2 \geq c_s^2$, and (100) would be contradicted. From this result and from (99) and (100), it finally follows that $c_2^2 > c_s^2$. Therefore of the two modes of wave motion for the same k and n, one has a subsonic phase velocity and the other a supersonic one. From (93) it can be seen that, for each value of k, c_2^2 (corresponding to the plus sign) increases whereas c_1^2 (corresponding to the minus sign) decreases with increasing n. That is to say, the supersonic wave speed increases whereas the subsonic wave speed decreases as the number of nodal planes are increased for each wavelength.

Eckart [1960, pp. 138–143] found the supersonic phase velocities but, because the terms containing g/c_s^2 were omitted by him, did not find the subsonic phase velocities (for each k) other than the one for free-surface waves, which are ruled out here by the rigidity of the upper boundary, but will be discussed in the next section. Thus Eckart did not find the internal gravity waves discussed above, or the duality of c^2 for each k and n.

That the phase velocity for the acoustic modes can be greater than the speed of sound seems already a little strange at first sight. That it should increase with the number of nodal planes in the fluid certainly calls for an explanation. The occurrence of nodal planes and of the interspersing maxima and minima of the displacement ζ in the interval $0 \leq z \leq d$ gives rise to a pattern of high and low values of ζ as shown in Fig. 3. The dots indicate highs and the circles lows. (Since $w(z) = -ikc\zeta(z)$, the maxima and minima of ζ are those of w, as far as their variations with z are concerned.) Now this pattern is similar to that shown in Fig. 1 for three-dimensional waves, and the increase of the phase velocity with the number of nodal planes can be explained by the argument given in Section 4. To clarify this situation further, consider a very large value of n in (92), and the corresponding acoustic mode. Equation (92) or (93) gives, with the positive sign taken,

$$c^2 = \left(\frac{n^2\pi^2}{k^2 d^2} + 1\right)c_s^2 = \frac{\lambda^2 + (2d/n)^2}{(2d/n)^2} c_s^2. \tag{101}$$

But λ corresponds to λ_k in Section 4, and $2d/n$ corresponds to λ_l. Thus

$$\frac{\lambda^2 + (2d/n)^2}{(2d/n)^2}$$

is $\csc^2 \theta$, with the θ given in (58), and (101) is exactly (61), after the $\sin \theta$ in (61) has been transposed and the equation squared. That c_s corresponds to c' is very important. It shows that for predominantly acoustic waves the plane waves still have their speed equal to the speed of sound, and the fact that c is greater than c_s is a direct result of reflection of the generic simple plane waves from the rigid boundaries. The argument presented here does not apply to waves of the gravity type, precisely because the simple plane waves do not exist for gravity waves.

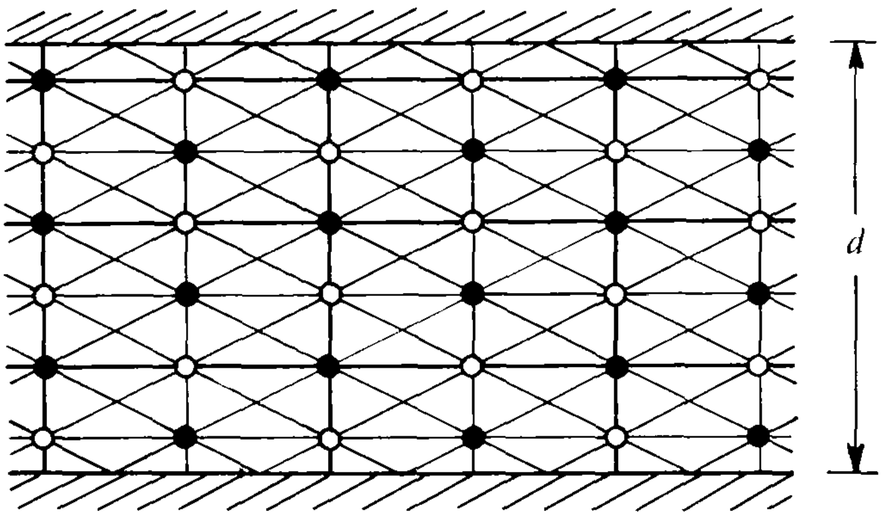

FIGURE 3. The high and low points in wave motion of the acoustic type, with several nodal planes between the horizontal boundaries. The vertical lines are crests (heavy) or troughs (light) due to the periodic variation with x; the horizontal ones mark the maxima (heavy) and minima (light) of the displacement. Slant lines are plane wave fronts. This pattern is similar to that shown in Fig. 1. Note that at the intersection of a light vertical line with a light horizontal line the actual displacement is positive and a maximum, because the product of two negative numbers is positive.

For a given c_s and a given n, c^2 decreases as k^2 increases. This can be proved by differentiating (92) with respect to k^2. The result is

$$\left(\frac{8k^2c^2}{c_s^2} - \beta'^2\right)\frac{dc^2}{dk^2} = 4c^2\left(1 - \frac{c^2}{c_s^2}\right),$$

or, after multiplication by c^2 and using (92) once more,

$$\left[\beta'^2c^2 - 8\left(\beta g - \frac{g^2}{c_s^2}\right)\right]\frac{dc^2}{dk^2} = 4c^4\left(1 - \frac{c^2}{c_s^2}\right). \tag{102}$$

Since

$$\frac{\beta'^2c_s^2}{4k^2} = c_1^2 + c_2^2 \quad \text{and} \quad c_1^2c_2^2 = \frac{c_s^2}{k^2}\left(\beta g - \frac{g^2}{c_s^2}\right) = \frac{(\gamma - 1)g^2}{k^2},$$

(102) can be written as

$$\frac{4k^2}{c_s^2}[(c_1^2 + c_2^2)c^2 - 2c_1^2c_2^2]\frac{dc^2}{dk^2} = 4c^4\left(1 - \frac{c^2}{c_s^2}\right). \tag{103}$$

Now for $c = c_1$, the coefficient of dc^2/dk^2 is negative and the right-hand side of (103) is positive. For $c = c_2$ the reverse is true. In either case

$$\frac{dc^2}{dk^2} < 0,$$

or, with c and k both understood to be positive,

$$\frac{dc}{dk} < 0. \tag{104}$$

This is to say that the longer the wavelength the faster the speed of propagation, for both subsonic and supersonic modes—a result not proved before.

10. WAVE MOTION IN AN ISOTHERMAL ATMOSPHERE WITH A FREE SURFACE

Since an isothermal atmosphere is infinite in extent if not bounded by an upper rigid boundary, a free surface can only be situated at infinity. However, for simplicity, a free surface at a finite height d will be considered. It is hoped that the result will shed some light on the case of infinite depth, and that at any rate if d is great (in comparison with the wavelength or any other reference length) the result should differ little from the result of an infinite atmosphere, especially for small values of z.

The solution of (86), which governs the motion, is still given by (87), (89), and (90). But the boundary condition

$$w'(d) = \frac{g}{c^2}w(d) \tag{105}$$

now yields the secular equation

$$q \coth\frac{q}{2} = \frac{2gd}{c^2} - \beta d, \tag{106}$$

in which

$$q^2 = \left(\beta^2 + 4k_i^2 - \frac{4\beta g}{c^2}\right)d^2$$

$$= \beta^2d^2 - \frac{4g\beta d^2}{c^2}\left(1 - \frac{1}{\gamma}\right) + 4k^2d^2\left(1 - \frac{c^2}{c_s^2}\right). \tag{107}$$

The theorem to be proved is that c^2 decreases as k^2 increases. The proof seems somewhat involved, although it is really quite straightforward.

First, the variation of q^2 with c^2 for a fixed positive k^2 can be ascertained from (107). For very small and very large values of c^2, q^2 is negative. When it is negative (106) takes the form

$$|q| \cot \left|\frac{q}{2}\right| = \frac{2gd}{c^2} - \beta d, \tag{108}$$

as can also be verified directly from (87), (89), (90), and (105). But there are values of c^2 for which q^2 is positive. In fact c_s^2 is such a value. For $c^2 = c_s^2$,

$$q^2 = \beta d^2 \left[\beta - \frac{4g(\gamma - 1)}{\gamma c_s^2}\right]. \tag{109}$$

Since

$$\beta = \frac{g}{R\bar{T}} = \frac{\gamma g}{c_s^2} \quad \text{and} \quad \gamma = \frac{N + 2}{N} \qquad (N = \text{degree of freedom of the gas molecule}), \tag{110}$$

it follows that

$$q^2 = \beta^2 d^2 \left[1 - \frac{4(\gamma - 1)}{\gamma^2}\right] = \left[\frac{\beta d(N - 2)}{N + 2}\right]^2 > 0. \tag{111}$$

For $c^2 = c_s^2$, the right-hand side of (106) has the value

$$\beta d\left(\frac{2}{\gamma} - 1\right) = \beta d \frac{N - 2}{N + 2} = q.$$

Hence for this particular value of c^2 the left-hand side of (106) is greater than the right-hand side. The maximum value of q^2 can be found from

$$0 = \frac{dq^2}{dc^2} = 4d^2 \left[\frac{g\beta(\gamma - 1)}{\gamma c^4} - \frac{k^2}{c_s^2}\right],$$

giving

$$c^2 = \left[\frac{g\beta(\gamma - 1)}{\gamma}\right]^{1/2} \frac{c_s}{k}. \tag{112}$$

The second derivative of q^2 with respect to c^2 is always negative. Hence the value for c^2 given in (112) corresponds to a maximum value of q^2, which is positive, since q^2 is positive for $c^2 = c_s^2$. The curve representing q^2 as a function of c^2 then assumes the form shown in Fig. 4. If

$$L = \begin{cases} q \coth \dfrac{q}{2} & \text{for} \quad q^2 > 0, \\[2ex] |q| \cot \left|\dfrac{q}{2}\right| & \text{for} \quad q^2 < 0, \end{cases} \tag{113}$$

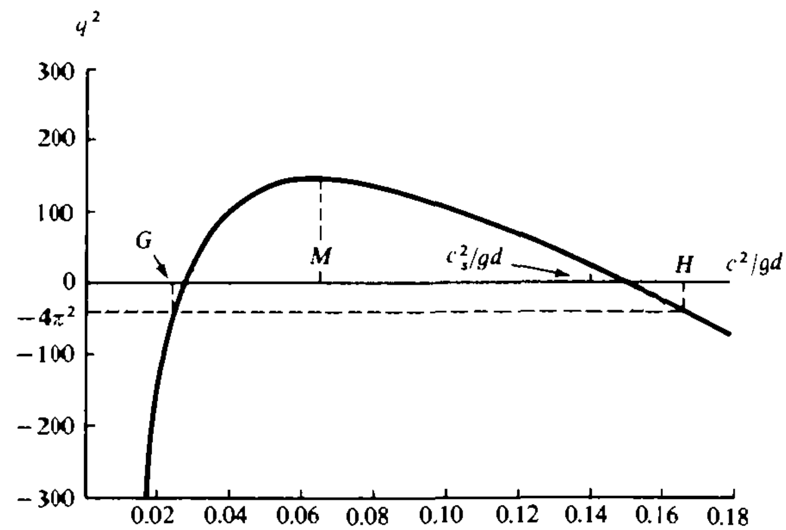

FIGURE 4. Plot of q^2 against c^2/gd according to (107). $\beta d = 10$, $\gamma = 1.4$, $kd = 10$, and $c_s^2/gd = 0.14$.

the curve representing L is shown in Fig. 5. As $c^2 \to 0$, there are infinitely many branches of the curve. The same is true as $c^2 \to \infty$. The right-hand side of (106) is also shown in the same figure, with the (single) curve marked R. (The unmarked branches all describe L.) The eigenvalues for c^2 are the abscissas of the points of intersection of the curves for L and R. Subsonic wave

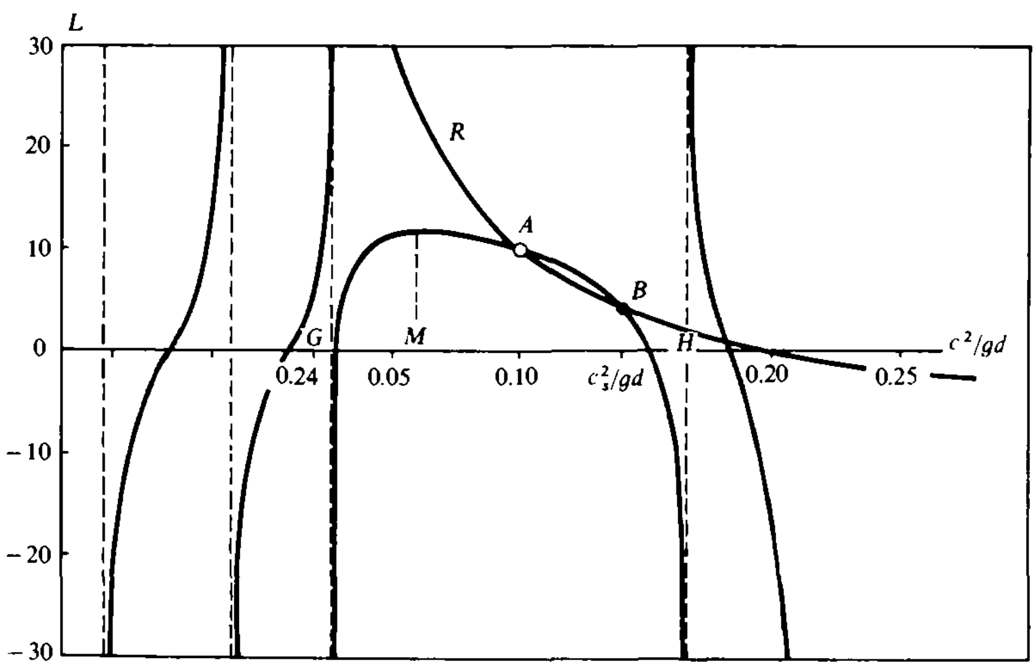

FIGURE 5. Plot of (108) with $\beta d = 10$. The horizontal scale changes at G, for the convenience of plotting the curves to its left. In this figure, the abscissa of B is only slightly greater than c_s^2/gd.

speeds are indicated by circles and supersonic speeds are indicated by dots at these points of intersection. Since

$$\begin{aligned} \frac{dL}{dq} &= \coth\frac{q}{2} - q\,\mathrm{csch}^2\frac{q}{2} > 0 \qquad \text{for} \qquad q^2 > 0, \\ \frac{dL}{d|q|} &= \cot\left|\frac{q}{2}\right| - |q|\csc^2\left|\frac{q}{2}\right| < 0 \qquad \text{for} \qquad q^2 < 0, \end{aligned} \tag{114}$$

it follows that dL/dq^2 is always positive. Hence L is a monotonically increasing function to the left of the point M (where q^2 has a maximum) and a monotonically decreasing function to the right of M. There are therefore two intersections of curve R with the center branch of curve L (q^2 positive) and only one intersection with curve R for each of the other branches of curve L. All the points of intersection corresponding to subsonic wave speeds correspond to positive L or R. The dot-points may be above or below the axis for c^2, but this is of no importance here.

Now if k^2 is increased, q^2 will be increased for $c^2 < c_s^2$ and decreased for $c^2 > c_s^2$, as can be seen from (107). Since dL/dq^2 is always positive, the branches of L to the left of the point $c^2 = c_s^2$ will move upward whereas those to the right will move downward. In either case the points of intersections will shift to the left and the eigenvalues of c^2 will be decreased. The theorem is therefore proved.

The eigenvalues for c^{-2} corresponding to subsonic wave speeds may be denoted by $\lambda_{-1}, \lambda_{-2}, \ldots$, and those corresponding to supersonic wave speeds by $\lambda_1, \lambda_2, \ldots$. Since λ_{-1} and λ_1 correspond to positive q^2, they correspond to free-surface waves, with the pertaining eigenfunctions (for w) free from zeros in $0 < z \leq d$. For λ_{-2} and λ_2, q^2 is between $-\pi^2$ and $-4\pi^2$, and the eigenfunction (which is not the same for λ_{-2} as for λ_2) given by (87),

$$w = 2Ae^{\beta z/2}\sinh q\left(\frac{z}{d}\right), \tag{115}$$

will vanish once in $0 < z < d$. For n equal to any positive integer, λ_{-n} and λ_n correspond to eigenfunctions which have exactly n zeros in $0 \leq z \leq d$. Therefore, for fixed k^2, subsonic wave speeds decrease as the number of nodal planes is increased, whereas the reverse is true for supersonic wave speeds.

Since the group velocity is

$$\frac{d(kc)}{dk} = c + k\frac{dc}{dk},$$

the conclusion that dc/dk is negative is tantamount to the conclusion that the group velocity is less than the phase velocity, since c and k can be taken to be positive for convenience. This conclusion is true for incompressible fluids and for an isothermal gas. It is probably true in general for gravity waves.

11. ESTIMATE OF PHASE VELOCITY

With reference to (69), Sturm's oscillation theorem states that the oscillation of the function f is more rapid for a smaller K (assumed positive) and a smaller G. For an incompressible fluid, the motion of which is governed by System I, the boundary condition $w(0) = 0$ demands that one zero be always situated at $z = 0$. Then Sturm's theorem states that the number of zeros for w cannot decrease if $\bar{\rho}$ and $[k^2\bar{\rho} + (g/c^2)\bar{\rho}']$ decrease. Let the lowest density in the range $0 \leq z \leq d$ be denoted by a and the highest density in the same range by b, and the algebraically least and greatest values of the mean density gradient be denoted by $-a'$ and $-b'$. (Note: a and a' do not necessarily occur at the same z. Nor do b and b'.) Comparing the zeros of w for the actual density stratification with those of f governed by the equation

$$(af')' - \left(k^2 a - \frac{g}{c^2} a'\right) f = 0,$$

we see that f oscillates faster than w. Similarly, w oscillates faster than h governed by

$$(bh')' - \left(k^2 b - \frac{g}{c^2} b'\right) h = 0.$$

Now, if the boundary conditions are

$$w(0) = 0, \qquad w(d) = 0,$$

then for n zeros in the interval $0 \leq z < d$, or $n + 1$ zeros in the interval $0 \leq z \leq d$,

$$\frac{ga'}{c^2 a} - k^2 \geq \left(\frac{n\pi}{d}\right)^2, \tag{116}$$

for otherwise even f could not have $n + 1$ zeros in that interval. Similarly, for the same number of zeros,

$$\frac{gb'}{c^2 b} - k^2 \leq \left(\frac{n\pi}{d}\right)^2. \tag{117}$$

For otherwise even h would have at least $n + 1$ zeros in the interval, and w would have at least $n+2$ zeros since $w(d) = 0$. Inequalities (116) and (117) can be combined to give a range in which c^2 must lie:

$$\frac{ga'd^2}{a(n^2\pi^2 + k^2d^2)} \geq c^2 \geq \frac{gb'd^2}{b(n^2\pi^2 + k^2d^2)}. \tag{118}$$

If the upper boundary is not fixed but free, $w \neq 0$ at $z = d$. Then for n zeros in the interval $0 \leq z < d$ (or the interval $0 \leq z \leq d$),

$$\frac{ga'}{c^2 a} - k^2 > \frac{(n-1)^2\pi^2}{d^2}, \tag{119}$$

for otherwise even f could not have n zeros in the half-open interval. Similarly

$$\frac{gb'}{c^2 b} - k^2 < \left(\frac{n\pi}{d}\right)^2. \tag{120}$$

For otherwise even h would have $n + 1$ zeros in the half open interval. Inequalities (119) and (120) can be combined to give a range for c^2 for the case in which a free surface is present:

$$\frac{ga'd^2}{a[(n-1)^2\pi^2 + k^2d^2]} > c^2 > \frac{gb'd^2}{b(n^2\pi^2 + k^2d^2)}. \tag{121}$$

For an isothermal gas, which is stratified in entropy, the eigenvalues of c^2 are given exactly in (93), and no estimate is necessary. For an arbitrarily stratified atmosphere, the system is not exactly a Sturm-Liouville system, because the parameter c^2 appears in several places, as in (28), and Sturm's theorems cannot be applied to obtain the ranges in which c^2 must lie.

Inequality (121) is valid not merely for a single layer of continuously stratified fluid, but also for one of many superposed continuously stratified layers, with density discontinuities at the interfaces and at the free surface, so long as the zeros under consideration all occur in that layer.

12. WAVE MOTION IN A STRATIFIED LIQUID WITH DENSITY DISCONTINUITIES

Sturm's theorems, which we have relied upon to obtain most of the general results concerning the phase velocity of waves propagating in a stratified liquid, are based on the continuity of the coefficients of the differential equation. For wave motion in a stratified liquid with density discontinuities, we can consider the governing differential system to be a system of differential equations with continuous coefficients and linked by interlocking boundary conditions, or a Sturm-Liouville system with discontinuous coefficients. To make the results obtained for a continuously stratified liquid applicable to a liquid with density jumps, Sturm's theorems must be generalized.

Since the boundary conditions are natural ones in the sense that they can be obtained by integrating the differential equation across the density discontinuities in the Stieltjes sense, these discontinuities are not likely to invalidate Sturm's theorems and the many results obtained from them for a continuously stratified liquid. We can consider the discontinuous density distribution to be the limit of a sequence of continuous density distributions with large gradients near the actual discontinuities, and attempt to show that the sets of eigenvalues for the sequence converge to a limit set, which is the set of eigenvalues for the discontinuous density distribution. It then follows that many properties possessed by the converging sets are also possessed by their limit. Alternatively, we can take the discontinuous density distribution as it is, and deal with the interlocking differential system directly, using one differential equation for each layer within which the density is continuous.

The density jump at the ith interface will be denoted by $(\Delta\rho)_i$, with i ranging from 1 to M if there are M surfaces of density discontinuity. A free surface is a special interface with the density above it equal to zero. Further discussion in this section can best be carried out in subsections.

12.1. *The Reality of c^2*

The condition at each interface is (36), with $l = 0$,

$$(\bar{\rho}w')_u - (\bar{\rho}w')_l + \frac{g}{c^2}\Delta\bar{\rho}\, w = 0, \tag{122}$$

in which

$$\Delta\bar{\rho} = \bar{\rho}_l - \bar{\rho}_u,$$

with the subscripts l and u indicating the lower and upper fluids, respectively. At a free surface, $\bar{\rho}_u = 0$, $\Delta\bar{\rho} = \bar{\rho}_l$, and

$$w'(d) = \frac{g}{c^2}w(d), \tag{123}$$

as obtained before. The differential equation for each layer is given in System I. Multiplying the differential equations by w^*, the complex conjugate of w, integrating between 0 and d layer by layer in the ordinary or Riemannian sense, and applying the boundary conditions at $z = 0$, at the interfaces, and at $z = d$, we have

$$\frac{g}{c^2}\sum_{i=1}^{M}(\Delta\rho)_i|w|^2 - \int_0^d \bar{\rho}|w'|^2\,dz - k^2\int_0^d \bar{\rho}|w|^2\,dz$$
$$-\frac{g}{c^2}\int_0^d \bar{\rho}'|w|^2\,dz = 0. \tag{124}$$

The last integral is an abbreviation for the sum of the integrals over the M (if there is a free surface) or $M + 1$ (if the upper boundary is rigid) layers, that is, it is an integral with the locations of the discontinuities excluded. From (124) it can be seen that c^2 is real. Furthermore, if $\bar{\rho}'$ is negative throughout and $(\Delta\rho)_i$ is positive for each i, c^2 is positive, and in the opposite case c^2 is negative. Thus, if c is the wave velocity (taken as positive) for a stable stratification, the amplification rate σ is kc if the stratification is reversed, or if the direction of gravity is reversed. In case the stratification is partly stable and partly unstable, some eigenvalues of c^2 can be expected to be positive and some negative.

12.2. *Equipartition of Energy*

Equation (124) also contains the theorem of the equipartition of energy. After division by k^2, it can be written in the more compact form

$$\frac{1}{k^2}\int_0^d \bar{\rho}|w'|^2\,dz + \int_0^d \bar{\rho}|w|^2\,dz = -\frac{g}{c^2k^2}\int_0^d \bar{\rho}'|w|^2\,dz, \tag{125}$$

in which the last integral is now in the Stieltjes sense. Since for an incompressible fluid in two-dimensional motion $c_s = \infty$ and $l = 0$, (27) assumes the form

$$iku = -w'. \tag{126}$$

Hence

$$|w'|^2 = k^2|u^2|,$$

and the left-hand side of (125) is obviously twice the kinetic energy of the fluid per unit width and per unit length. Equation (38) enables us to write the right-hand side of (125) in the form

$$-g\int_0^a \bar{\rho}'\zeta_0^2\,dz$$

in the Stieltjes sense. This is twice the potential energy per unit width and per unit length. (The liquid in the troughs is raised to the crests, each element $\zeta\,dx$ by the height ζ. Thus the total potential energy is proportional to the integral of ζ^2 over a *half* wavelength. The integrals should be considered to have been extended over a wavelength, and then divided by that wavelength.) Thus the total energy of the wave motion is equally partitioned—one half in the form of kinetic energy, the other half in the form of potential energy. This result is, of course, also valid if no density discontinuities are present.

As has been seen in Section 7, there are infinitely many eigenvalues of c^2 for an assigned value of k, each corresponding to a different mode characterized by the number of zeros of w in the interval $0 \leq z < d$. It will now be shown that the potential and kinetic energies of the various modes with the same wave number are entirely separable—within the framework of a linear theory.

First, consider the simple case of a continuously stratified liquid confined between two rigid boundaries. Let $\lambda = c^{-2}$ and let w_r be associated with λ_r and w_s with λ_s. Equation (31) can be written

$$(\bar{\rho}w')' - (k^2\bar{\rho} + \lambda g\bar{\rho}')w = 0. \tag{127}$$

Multiplying

$$(\bar{\rho}w_r')' - (k^2\bar{\rho} + \lambda_r g\bar{\rho}')w_r = 0 \tag{127a}$$

by w_s and

$$(\bar{\rho}w_s')' - (k^2\bar{\rho} + \lambda_s g\bar{\rho}')w_s = 0 \tag{127b}$$

by w_r, subtracting the results, integrating (by parts if necessary) between $z = 0$ and $z = d$, and utilizing the boundary conditions that w vanishes at the end points, we have

$$\int_0^d \bar{\rho}' w_r w_s \, dz = 0. \tag{128}$$

This establishes the orthogonality of the modes for λ_r and λ_s, and the separability of potential energies of the any two modes for the same k. With this demonstrated, we can multiply (127a) by w_s and integrate the results, producing

$$\int_0^d \bar{\rho}(w_r' w_s' + k^2 w_r w_s) \, dz = 0. \tag{129}$$

Using (126), we can rewrite (129) in the form

$$k^2 \int_0^d \bar{\rho}(w_r w_s - u_r u_s) \, dz = 0,$$

or

$$\frac{1}{2}\int_0^d \bar{\rho}(2w_r w_s + u_r u_s^* + u_s u_r^*) \, dz = 0. \tag{129a}$$

(Remember that if w is taken to be real, u is purely imaginary.) Now (129a) certainly means that

$$\frac{1}{2}\int_0^d \bar{\rho}[(w_r + w_s)^2 + (u_r + u_s)(u_r^* + u_s^*)] \, dz$$
$$= \frac{1}{2}\int_0^d \bar{\rho}[w_r^2 + w_s^2 + |u_r|^2 + |u_s|^2] \, dz, \tag{130}$$

which means that the kinetic energy is separable.

If there are surfaces of density discontinuity, the same approach, with the proper application of the interfacial conditions (and the free-surface condition if a free surface exists), gives

$$\sum_{i=1}^{M} (\Delta \bar{\rho} \, w_r w_s)_i + \int_0^d \bar{\rho}' w_r w_s \, dz = 0, \tag{128a}$$

in which the integral is in the Riemannian sense with the points of discontinuity in density excluded. (The integral alone would suffice if it were in the Stieltjes sense.) Thus the potential energy is again separable. With (128a) and (129a), Eq. (130) is again obtained, and the separability of the kinetic energy is again established.

It remains to mention that if the wave numbers are different, the net coupling effect is finite over a distance in the x-direction, however long, for all the modes pertaining to these wave numbers, and is therefore zero per unit distance in the limit.

The equipartition of energy is valid also for three-dimensional waves. The proof is similar and need not be reproduced here.

The foregoing development is for an incompressible fluid. The case of a compressible fluid is more complicated, since the internal energy is also involved. No simple result concerning the energy partition is known.

12.3. *Existence of Eigenvalues*

To prove the existence of eigenvalues, consider Eqs. (74) and (76), with the K's and G's satisfying (78). We shall identify K_1 with $\bar{\rho}_1$, K_2 with $\bar{\rho}_2$, G_1 with

$$k_1^2\bar{\rho}_1 + \frac{g\rho_1'}{c_1^2},$$

and G_2 with

$$k_2^2\bar{\rho}_2 + \frac{g\bar{\rho}_2'}{c_2^2}.$$

Hence $K_1 = K_2$, and $c_1^2 = c_2^2$ for the same $\bar{\rho}$ and k^2.

Let the upper boundary be rigid, and the depths of the layers be denoted by $d_1, d_2, d_3, \ldots, d_{M+1}$. The height of the interfaces will be denoted by

$$D_1 = d_1, \qquad D_2 = d_1 + d_2, \qquad \ldots \qquad D_M = d_1 + d_2 + \cdots + d_M.$$

From (74) and (76), we can deduce Picone's identity

$$\frac{d}{dz}\left[w_1^2\left(K_1\frac{w_1'}{w_1} - K_2\frac{w_2'}{w_2}\right)\right] = (G_1 - G_2)w_1^2 + (K_1 - K_2)w_1'^2 + K_2\left[w_1' - w_2'\frac{w_1}{w_2}\right]^2. \tag{131}$$

Now the interfacial conditions are given by (122), to be satisfied by w_1 and w_2, and w is continuous at the interfaces. Hence, if $w_1(0) = 0$ and $w_1(z_1) = 0$, with z_1 in the layer d_r, the sum of the r integrals of (131) over the lengths $d_1, d_2, \ldots, z_1 - D_{r-1}$ is, after utilization of (122),

$$g\sum_{i=1}^{r-1}\left[\left(\frac{\Delta\bar{\rho}_i}{c^2}\right)_1 - \left(\frac{\Delta\bar{\rho}_i}{c^2}\right)_2\right](w_1^2)_i = \sum_r\left[\int(G_1 - G_2)w_1^2\,dz + \int(K_1 - K_2)w_1'^2\,dz + \int K_2\left(w_1' - w_2'\frac{w_1}{w_2}\right)^2 dz\right], \tag{132}$$

provided $w_2 \neq 0$ in $0 < z < z_1$. (The cases $w_2(0) = 0$ and $w_2(d) = 0$ need not be excluded, since then $w_1/w_2 = w_1'/w_2'$ at $z = 0$ or d.) But under the assumptions $K_1 \geq K_2$, $G_1 \geq G_2$, the right-hand side is positive and nonzero, since $K_2 > 0$, and G_1 is not equal to G_2 everywhere in all considerations. Hence, if

$$\left(\frac{\Delta\bar{\rho}_i}{c^2}\right)_1 \leq \left(\frac{\Delta\bar{\rho}_i}{c^2}\right)_2,$$

w_2 must vanish at least once in the interval $0 < z < z_1$. *If w_1 has $n + 1$ zeros at $z = 0, z_1, z_2, \ldots$, and d, there are zeros of w_2 between these ordinates. This is a generalized version of Sturm's first comparison theorem.* From it two corollaries can be deduced:

COROLLARY 1. *For the same k and same number $(n + 1)$ of zeros in the closed interval $0 \leq z \leq d$, c^2 is smaller if $\bar{\rho}$ is increased by a constant or if $\bar{\rho}$ is increased everywhere while $|\bar{\rho}'|$ and $(\Delta\rho)_i$ are decreased, and greater if $\bar{\rho}$ is decreased everywhere while $|\bar{\rho}'|$ and $(\Delta\rho)_i$ are increased.*

COROLLARY 2. *For the same $\bar{\rho}$ and same number $(n + 1)$ of zeros in $0 \leq z \leq d$, c^2 is decreased if k^2 is increased.**

Evidently, if these corollaries were not true, w_2 would have $n + 2$ zeros in $0 \leq z \leq d$, two at $z = 0$ and $z = d$, and n between the zeros of $w_1(z)$.

Next, we show that if $K_1 \geq K_2$ and $G_1 \geq G_2$, and if

$$\left(\frac{\Delta\bar{\rho}_i}{c^2}\right)_1 \leq \left(\frac{\Delta\bar{\rho}_i}{c^2}\right)_2, \tag{133}$$

$$w_1(b) \neq 0, \qquad w_2(b) \neq 0, \tag{134}$$

then

$$\frac{K_1(b)\, w_1'(b)}{w_1(b)} > \frac{K_2(b)\, w_2'(b)}{w_2(b)}, \tag{135}$$

provided w_1 and w_2 have the same number of zeros in $0 \leq z < b$, and $w_1(0) = 0$, $w_2(0) = 0$. The point $z = b$ may be any point. But in the applications b will be identified with any of the D's or with the total depth d. (If it is identified with d, the Mth interface is a free surface located at $z = d$.) If again the point $z = b$ is in the layer d_r, integration of (131) between $z = z_b$ (the last zero of w_1 before b is reached) and $z = b$ produces

$$w_1^2\left[\frac{K_1(b)w_1'(b)}{w_1(b)} - \frac{K_2(b)w_2'(b)}{w_2(b)}\right] + \text{left-hand side of (132)}$$
$$= \text{right-hand side of (132)}, \tag{136}$$

since, by virtue of the generalized Sturm's theorem just obtained, $w_2 \neq 0$ in $z_b < z < b$. In obtaining (136), the boundary condition $w_1(z_b) = 0$ and the interfacial conditions between z_b and b have been utilized. Since

$$\frac{w_1}{w_2} = \frac{w_1'}{w_2'} \qquad \text{at} \qquad z = 0,$$

* This result was presented in a paper [Yih, 1962b] to the Conference on Fluid Dynamics in Geophysics held in Boulder, Colorado, in September 1962. The same result is reached by Yanowitch [1962] by using a variational approach.

no difficulty arises if z_b is zero. The conclusion follows from (136) directly and obviously. *Equation* (135) *obtained for discontinuous K and G, constitutes the generalized second theorem of Sturm.* From it one corollary can be deduced:

COROLLARY. *If the upper boundary is free, identify b with d. Then, for the same* $\bar{\rho}$ *and the same n* (*number of zeros in* $0 \leq z \leq d$), c^2 *is smaller if k is greater. For then* $K_1 = K_2$, $G_1 > G_2$, *and* (135) *and* $w'(d) = (g/c^2)w(d)$ *together give* $c_1^2 < c_2^2$.

We have shown, among other things, that whether or not the upper boundary is free, c^2 decreases and k increases. We shall now grapple with the main task of proving the existence of the eigenvalues, for a fixed k.

Consider first the case of two superposed continuously stratified layers confined between two rigid boundaries at $z = 0$ and $z = d$. Let the interface be situated at $z = d_1$. We can always manage to have $w(0) = 0$, $w(d) = 0$, and w continuous at $z = d_1$, by starting one part of the solution from $z = 0$ and the other from $z = d$, and by multiplying one of these by a constant to make w continuous at $z = d_1$, provided $w(d_1) \neq 0$ for either part of the solution. We shall make this provision, without restricting the validity of the proof. For if $w(d_1) = 0$, either $(\bar{\rho}w')_u \neq (\bar{\rho}w')_l$, in which case the interfacial condition is violated, or $(\bar{\rho}w')_u = (\bar{\rho}w')_l$, in which case at least one (the assumed one) eigenvalue exists.

With $w \neq 0$ at $z = d_1$ either for the lower or for the upper fluid, (135) with $b = d_1$ states that, for the same $\bar{\rho}$,

$$\frac{\bar{\rho}_l\, w_1'(d_1)}{w_1(d_1)} > \frac{\bar{\rho}_l\, w_2'(d_1)}{w_2(d_1)}, \tag{137}$$

if $c_1^2 > c_2^2$ (so that $G_1 > G_2$), since $K_1 = K_2 = \bar{\rho}$. In (137) $\bar{\rho}_l$ is the density of the lower fluid at $z = d_1$. Hence $\bar{\rho}_l\, w_l'/w$ decreases at $z = d_1$ as c^2 decreases.* In obtaining (135) or (137) we have integrated between $z = z_b$ and $z = b$ ($b = d_1$ in this application). If we integrate (131) between $z = z_b'$ and $z = d_1$, we get, if z_b' is the first zero of w_2 above $z = b$,

$$\frac{\bar{\rho}_u\, w_1'(d_1)}{w_1(d_1)} < \frac{\bar{\rho}_u\, w_2'(d_1)}{w_2(d_1)}, \tag{138}$$

as can be easily verified by reviewing the derivation of (135). Thus $\bar{\rho}_u w_u'/w$ increases at $z = d_1$ as c^2 decreases. Since w is continuous at $z = d_1$ whereas w' is not, we conclude that

$$\frac{\bar{\rho}_l w_l' - \bar{\rho}_u w_u'}{w} \tag{139}$$

* That w' in (137) is for the lower fluid is implied. We may make this clear by giving w' the subscript l. Similarly, the subscript u for w' is implied in (138).

decreases as c^2 decreases. Now as c^2 decreases, zeros of w_l and of w_u will appear (not in general for the same c^2) one after the other at $z = d_1$. Let these be taken together and the corresponding values of $\lambda = c^{-2}$ be arranged in ascending order of magnitude and be named λ_1, λ_2, λ_3, etc. Then as c^2 decreases from $1/\lambda_1$ to $1/\lambda_2$, the quantity (139) decreases. Barring the fortuitous case $(\rho w')_u \to (\rho w')_l$ as $w_l(=w_u) \to 0$, which means that the λ involved is an eigenvalue for both the upper and lower layers, and hence for the system as a whole, it must indeed decrease from $+\infty$ to $-\infty$. In the meantime $g\ \Delta\bar{\rho}/c^2$ increase as c^2 decreases. Hence at some value of c^2

$$(\bar{\rho}w')_l - (\bar{\rho}w')_u = \frac{g}{c^2}\,\Delta\bar{\rho}\ w.$$

which is the interfacial condition (122). Thus, eigenvalues exist. They *separate* the ensemble of eigenvalues for *either* layer considered as confined between rigid boundaries (at $z = 0$ and $z = d_1$, or $z = d_1$ and $z = d$), and, *a fortiori*, separate the eigenvalues of each layer, considered as so confined.

The result achieved can now be looked upon to mean that, if one starts from $w(0) = 0$ and $w'(0) = 1$, integrates step by step, and proceeds at the interface toward higher values of z with $w_u = w_l$ and w'_u calculated from (122), w will be zero at $z = d$ for infinitely many values of c^2. With this in mind, one can consider next a superposition of three layers, confined between the rigid boundaries $z = 0$ and $z = d$. Similar arguments lead to the existence of eigenvalues. Either interface can be taken to be the place where the satisfaction of (122) is to be established, while the conditions at the rigid boundaries and the other interface are satisfied in the procedure of calculating w from above and from below. As a corollary, the eigenvalues separate the ensemble of those for the layer d_1 and those for the two-layer system $d_2 + d_3$, with the interface $z = d_1$ considered as rigid. Or, alternatively, they separate those of the double layer of depth $d_1 + d_2$ and those of the layer of depth d_3, with the interface $z = d_1 + d_2$ considered as rigid. Extension of the result to apply to the case of four and more layers is obvious, and we consider the existence of the eigenvalues proved for the case of a rigid upper boundary.

If the upper boundary is free, the proof is quite similar. The case of one layer with no interface can be dealt with easily enough. The quantity w'/w at the free surface has to decrease according to (132), if c^2 decreases, whereas g/c^2 has to increase. Between two values of c^2 which make $w = 0$ at $z = d$ (and we know these to exist) w'/w decreases from $+\infty$ to $-\infty$. Hence for some c^2

$$w' = \frac{gw}{c^2},$$

which is the free surface condition. The case of two, three, and more layers (with one, two and more interfaces) can be dealt with similarly since we have just established that there are values of c^2 which make $w = 0$ at $z = d$, so that similar arguments apply. We have then taken care of the existence of the eigenvalues completely when density discontinuities are present in the fluid. In fact, we have even roughly located where these should lie.

12.4. *Simple Proofs for the Variations of c^2 and σ^2 with k^2*

We have obtained the result that c^2 decreases as k^2 increases by using the generalized second theorem of Sturm. But this classical approach, requiring fixing the number of zeros of the eigenfunction, is tedious. We shall present a simpler proof [Yih, 1974a] of the theorem that c^2 decreases as k^2 increases. The method was first used by Groen [1948] to show that σ^2 *increases* with k^2 for fixed boundaries and later by Heyna and Groen [1958] to extend that result to free surfaces.

Consider (31) with the rigid-boundary condition at the bottom,

$$w(0) = 0.$$

The interfacial condition is (42) with $l = 0$, i.e.,

$$(\bar{\rho}w')_l - (\bar{\rho}w')_u = \frac{1}{c^2}\Delta\rho\, gw,$$

where $\Delta\rho = \rho_l - \rho_u$. If the upper surface is rigid, the boundary condition there is

$$w(d) = 0.$$

If the upper surface is free, the condition there is (45). Let k^2 increase by dk^2 and c^2 by dc^2, and let the corresponding variation in w be ε. Then from (31), ε satisfies

$$(\bar{\rho}\varepsilon')' - \left(k^2\bar{\rho} + \frac{g\bar{\rho}'}{c^2}\right)\varepsilon = (dk^2)\bar{\rho}w - \frac{g\bar{\rho}'}{c^4}(dc^2)w. \tag{140}$$

The boundary conditions on ε are $\varepsilon = 0$ at a rigid surface and (by perturbation of the free-surface condition)

$$\varepsilon' = \frac{g}{c^2}\varepsilon - \frac{g(dc^2)}{c^4}w$$

at a free surface. The conditions on ε at an internal surface of discontinuity can be similarly obtained by perturbation. We now multiply (31) by ε and (140) by w, integrate (layer by layer if there are density discontinuities) between 0 and d, and apply the boundary and interfacial conditions on w and ε. The difference of the two equations so obtained, with the last integral understood to be in the Stieltjes sense if there are any density discontinuities, is

$$0 = (dk^2)\int_0^d \bar{\rho} w^2\, dz - \frac{g(dc^2)}{c^4}\int_0^{d+} \bar{\rho}' w^2\, dz,$$

in which $d+$ means "just above $z = d$." Since the first integral in the above equation is positive definite and the second negative definite, we derive from it that

$$\frac{dc^2}{dk^2} < 0. \qquad (141)$$

Note that this result has been obtained by perturbation, which does not change the mode, so that the result is automatically for the same mode and for the same mode only. Herein resides the possibility of the simple approach.

To investigate the variation of σ^2 ($\sigma = kc$) with k^2, we write (31) in the form

$$(\bar{\rho} w')' - k^2\left(\bar{\rho} + \frac{g\bar{\rho}'}{\sigma^2}\right)w = 0,$$

the interfacial conditions in the form

$$(\bar{\rho} w')_l - (\bar{\rho} w')_u = \frac{k^2}{\sigma^2}\,\Delta\rho\, g w,$$

and the free-surface condition (if there is a free surface) in the form

$$w' = \frac{k^2}{\sigma^2}\, g w.$$

The condition at any rigid boundary remains the same. Multiplying the differential equation at the beginning of this paragraph by w, integrating between 0 and d, layer by layer if there are surfaces of density discontinuities, and using the boundary conditions and interfacial conditions, we find that

$$\int_0^{d+}\left(\bar{\rho} + \frac{g\bar{\rho}'}{\sigma^2}\right)w^2\, dz < 0.$$

The equation corresponding to (140) now has the form

$$(\bar{\rho} \varepsilon')' - k^2\left(\bar{\rho} + \frac{g\bar{\rho}'}{\sigma^2}\right)\varepsilon = (dk^2)\left(\bar{\rho} + \frac{g\bar{\rho}'}{\sigma^2}\right)w - \frac{k^2 g\bar{\rho}'}{\sigma^4}(d\sigma^2)w. \qquad (140a)$$

We can then proceed as we did for the variation of c^2 with k^2, reaching the conclusion [Groen, 1948; Heyna and Groen, 1958] that

$$\frac{d\sigma^2}{dk^2} > 0. \qquad (141a)$$

If we choose k and c to be positive, so that σ is also positive, we can write the two main results (141) and (141a) as

$$\frac{dc}{dk} < 0 \quad \text{and} \quad \frac{d\sigma}{dk} = c + \frac{dc}{dk}k > 0,$$

similar arguments apply. We have then taken care of the existence of the eigenvalues completely when density discontinuities are present in the fluid. In fact, we have even roughly located where these should lie.

12.4. *Simple Proofs for the Variations of c^2 and σ^2 with k^2*

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Consider (31) with the rigid-boundary condition at the bottom,

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The boundary conditions on ε are $\varepsilon = 0$ at a rigid surface and (by perturbation of the free-surface condition)

$$\varepsilon' = \frac{g}{c^2}\,\varepsilon - \frac{g(dc^2)}{c^4}\,w$$

at a free surface. The conditions on ε at an internal surface of discontinuity can be similarly obtained by perturbation. We now multiply (31) by ε and (140) by w, integrate (layer by layer if there are density discontinuities) between 0 and d, and apply the boundary and interfacial conditions on w and ε. The difference of the two equations so obtained, with the last integral understood to be in the Stieltjes sense if there are any density discontinuities, is

$$0 = (dk^2)\int_0^d \bar{\rho}w^2\,dz - \frac{g(dc^2)}{c^4}\int_0^{d+} \bar{\rho}'w^2\,dz,$$

in which $d+$ means "just above $z = d$." Since the first integral in the above equation is positive definite and the second negative definite, we derive from it that

$$\frac{dc^2}{dk^2} < 0. \tag{141}$$

Note that this result has been obtained by perturbation, which does not change the mode, so that the result is automatically for the same mode and for the same mode only. Herein resides the possibility of the simple approach.

To investigate the variation of σ^2 ($\sigma = kc$) with k^2, we write (31) in the form

$$(\bar{\rho}w')' - k^2\left(\bar{\rho} + \frac{g\bar{\rho}'}{\sigma^2}\right)w = 0,$$

the interfacial conditions in the form

$$(\bar{\rho}w')_l - (\bar{\rho}w')_u = \frac{k^2}{\sigma^2}\,\Delta\rho\, gw,$$

and the free-surface condition (if there is a free surface) in the form

$$w' = \frac{k^2}{\sigma^2}\,gw.$$

The condition at any rigid boundary remains the same. Multiplying the differential equation at the beginning of this paragraph by w, integrating between 0 and d, layer by layer if there are surfaces of density discontinuities, and using the boundary conditions and interfacial conditions, we find that

$$\int_0^{d+}\left(\bar{\rho} + \frac{g\bar{\rho}'}{\sigma^2}\right)w^2\,dz < 0.$$

The equation corresponding to (140) now has the form

$$(\bar{\rho}\varepsilon')' - k^2\left(\bar{\rho} + \frac{g\bar{\rho}'}{\sigma^2}\right)\varepsilon = (dk^2)\left(\bar{\rho} + \frac{g\bar{\rho}'}{\sigma^2}\right)w - \frac{k^2 g\bar{\rho}'}{\sigma^4}\,(d\sigma^2)w. \tag{140a}$$

We can then proceed as we did for the variation of c^2 with k^2, reaching the conclusion [Groen, 1948; Heyna and Groen, 1958] that

$$\frac{d\sigma^2}{dk^2} > 0. \tag{141a}$$

If we choose k and c to be positive, so that σ is also positive, we can write the two main results (141) and (141a) as

$$\frac{dc}{dk} < 0 \quad\text{and}\quad \frac{d\sigma}{dk} = c + \frac{dc}{dk}k > 0,$$

the latter of which means that the group velocity c_g, which is equal to $d\sigma/dk$, is positive. More generally, (141a) means that c and c_g are in the same direction. Since c decreases as k increases, we see that

$$c_g < c,$$

which implies that there are no upstream waves ahead of an obstacle placed in a stream of a stratified fluid of uniform velocity, or moving with that velocity in a stratified fluid at rest, even if that velocity is less than the greatest velocity of waves that can propagate in the stratified fluid at rest. See a discussion of this point at the end of Section 16. Also, a discussion of the group velocity of waves propagating in two or three dimensions will be given in Section 13.

12.5. *Surface of Density Discontinuity*

In Lamb's book [1945, p. 371] detailed calculations are given for the phase velocity of waves propagating in two superposed layers of homogeneous fluids with different densities. There is a density discontinuity at the surface of contact (the interface) and a much greater one at the free surface, where the density decreases abruptly from that of the upper fluid to zero. The calculations show that for each wave number k there are two values of c^2. For the greater value there is no zero of w in the range $0 \leq z \leq d$ other than the one at $z = 0$. For the smaller value the situation is quite different. There is a zero of w situated in the upper layer. The wave motion is largely interfacial, and the free surface is practically flat.

The result mentioned in the preceding paragraph suggests that a surface of discontinuity remains nearly flat for wave modes with corresponding values of c^2 which are much smaller than the value of c^2 pertinent to that surface. The association of a certain eigenvalue for c^2 with a certain surface of density discontinuity is not an ambiguous procedure, at least in principle. First of all, we can separate, at least in principle, those eigenvalues which correspond to density discontinuities from those corresponding to continuous density stratifications by letting the continuous density variations approach zero. If there are N surfaces of discontinuities there will be N eigenvalues of c^2 which change but little as the density gradient in each layer becomes smaller and smaller, and which approach their N limits as the density gradient in each layer approaches zero. These N eigenvalues clearly correspond to the N surfaces of density discontinuity. The attribution of each of these eigenvalues to a particular surface of discontinuity is possible if the eigenvalues of c^2 are known as functions of the density differences $\Delta\rho_i$ ($i = 1, 2, 3, \ldots, N$); for if $(c^2)_i$ vanishes with $\Delta\rho_i$ (and one of the N eigenvalues for c^2 will vanish with $\Delta\rho_i$), then it belongs to the surface at which the density discontinuity is $\Delta\rho_i$.

To prove that a free surface or interface will remain nearly flat for wave modes with much smaller values of c^2 than the one belonging to it, we can make use of (118). First, consider a high enough mode for which there are two zeros of w in a layer of depth d_1, in which the maximum numerical value of the density gradient is a' and the smallest density is a. Then (118) states that

$$c^2 \leq \frac{ga'd_1^2}{a(\pi^2 + k^2 d_1^2)}. \tag{142}$$

Since the differential system governing wave motion is linear and homogeneous, the dependent variable w can be multiplied by an arbitrary constant. We shall assume that in general $w = 0(1)$. Then, for the $n + 1$ zeros of w in $0 \leq z \leq d$, $w' = O(n)$. Thus (122) shows that, at most,

$$w = O\left[\frac{a'nd_1^2}{a(\pi^2 + k^2 d_1^2)}\right] \tag{143}$$

at any surface of density discontinuity. Thus w and, with it, ζ, are very small for small a' or large k, and therefore the surface is very flat for correspondingly small c^2. If there are $n_1 + 1$ zeros in the depth d_1, (142) is replaced by

$$c^2 \leq \frac{ga'd_1^2}{a(n_1^2\pi^2 + k^2 d_1^2)},$$

and (143) by

$$w = O\left[\frac{a'nd_1^2}{a(n_1^2\pi^2 + k^2 d_1^2)}\right].$$

Hence as n_1 increases, c^2 eventually decreases, and, with it, w and ζ eventually decrease, provided $n/n_1^2 \to 0$, as is in general the case. Hence for small density gradient, or large wave number k, or high wave modes for each k, surfaces of density discontinuity will remain nearly flat.

Next, consider interfacial waves which leave some surfaces of density discontinuity nearly flat. If the interfacial wave belongs to the ith interface, then $c^2 = O(\Delta\rho_i)$ and is very small if $\Delta\rho_i$ is very small. (These statements can be tested by finding the eigenvalues of c^2 for superposed homogeneous layers with different densities.) At any other surface (jth, say) of discontinuity with $\Delta\rho_j \gg \Delta\rho_i$, $w = O(\Delta\rho_i/\Delta\rho_j)$, as can be seen from (122) applied at the jth discontinuity. With the depths of the layers fixed, and if $\Delta\rho_i$ is very small compared with the other density jumps, the interfacial waves belonging to $\Delta\rho_i$ will then leave the other interfaces and the free surface nearly flat.

These discussions lead to an important conclusion concerning the spectrum of c^2. If the density gradient in each layer is small and the depth of each layer is not great, so that c^2 for waves internal to a layer (that is, there are two zeros at least in some layers) is small compared to c^2 for interfacial waves, we can find the eigenvalues for c^2 first for waves propagating in homogeneous layers, each with a depth and a density respectively equal to the depth and the mean density in the corresponding layer. These will be the eigenvalues of c^2 for the interfacial waves and free-surface waves, because the small continuous density variation in each layer can only give rise to a small correction. Next the eigenvalues for waves internal to each layer are found, with the interfaces considered as rigid. We can then arrange all the eigenvalues found in a descending order of magnitude, and the phase-velocity spectrum is found.

A discussion of how the zeros appear as the eigenvalues of c^2 decrease for each value of k may be instructive. Consider first the simple case of two superposed layers of continuously stratified fluids, with a density discontinuity at the surface of contact, and bounded by rigid horizontal planes. For the first mode there are just two zeros of w, one at each rigid boundary. If the density gradient in each layer is slight, this mode certainly corresponds to the interfacial wave. For the next (and smaller) eigenvalue there is an additional zero in one of the layers, and if the density gradient in that layer is small, the interface will remain nearly flat, as it will for subsequent eigenvalues. The additional zeros may occur in either layer.

The next simplest case is the case with three superposed layers between two rigid boundaries, each continuously stratified in density. The first mode, corresponding to the largest eigenvalue of c^2, has two zeros of w, one at each rigid boundary. The next mode has an additional zero which must occur in the middle layer if the density gradient in each layer is slight, for otherwise the second mode would correspond to a truly internal wave (internal to a layer, as distinguished from an interfacial wave), whereas the phase velocity of an interfacial wave is greater than that of a truly internal wave if the density gradient in each layer is very small. The next mode will bring in a fourth zero, and in whatever layer it may be, some layers will have two zeros of w. For this mode and higher modes, the interfaces will remain nearly flat if indeed the eigenvalues of c^2 for truly internal waves are much smaller than those for interfacial waves.

The case of two superposed layers with a free surface is similar to the case just discussed. If the density gradient in each layer is small, the first two modes will correspond to the free-surface wave and the interfacial wave. For the third and higher modes the surfaces of density stratification will be nearly flat.

12.6. *The Case of Infinite Depth*

In the case of infinite depth, the most interesting fact from the physical point of view is that an inviscid nonhomogeneous fluid is capable of irrotational wave motion, in spite of the density variation, whatever the density variation may be. Let there be a free surface at $z = 0$, so that

$$w'(0) = \frac{g}{c^2}\, w(0),$$

and let $w \to 0$ as $z \to -\infty$. For an incompressible fluid the governing differential equation is (31). This equation and the boundary conditions are evidently satisfied by

$$w(z) = e^{kz}, \qquad c^2 = g/k, \tag{144}$$

whatever function $\bar{\rho}$ is of z. Thus w varies exponentially with kz, whereas its sinusoidal variation with kx has been assumed. Hence the velocity component w, and in fact also u, are harmonic functions, and the motion is irrotational.

That the solution should be independent of $\bar{\rho}$ and irrotational besides is certainly extraordinary. The explanation furnishes a striking example of the consequence of Bjerknes' theorem. As can be readily verified, $w' = kw$ everywhere. Equations (25), (31), and (126) together show that $\rho = -i\bar{\rho}'w/kc$. Since, according to (24) and (126), $p = c\bar{\rho}u$ and $u = (i/k)w'$, p and ρ are in phase (note that $\bar{\rho}'$ is negative). Hence isopycnic lines are also isobars. This fact can also be proved by adopting a frame of reference moving with the waves to achieve steadiness of flow, and then by showing that p is constant along a streamline, which in steady flows is also an isopycnic line. Thus **grad** p is parallel to **grad** ρ, and Bjerknes' theorem—Eq. (57) of Chapter 1—states that circulation will be preserved. If then the motion is irrotational at one instant, it will continue to be. But within the framework of a linear theory all standing waves have a moment of instantaneous rest, which is a state of irrotationality, and all progressive waves are superpositions of two standing wave systems. Hence so long as the circulation is preserved, waves of all lengths will be irrotational.

The solution (144) in fact is valid even for a compressible fluid, as can be easily verified on putting $l = 0$ in (28). The isobars are lines of constant entropy—and in a frame of reference moving with the waves are also the streamlines, as can be verified simply. The density and entropy stratifications are entirely immaterial, and the motion is again irrotational, furnishing another example of the consequence of Bjerknes' theorem.

So far only two-dimensional waves have been discussed. It is obvious that, within the framework of a linear theory, these can be superposed to form three-dimensional waves, which will still be irrotational and characterized by the coincidence of the isobars with the isopycnic surfaces or (in the case of a compressible fluid) with the surfaces of constant entropy. Explicitly, the solution is

$$w(z) = e^{k'z}, \qquad k' = (k^2 + l^2)^{1/2}, \qquad c^2 = \frac{gk'}{k^2}. \tag{145}$$

It can be verified readily that this solution satisfies (28), (30), and the boundary condition (35).

If the fluid is compressible and isothermal, so that c_s is constant, another curious solution is

$$w(z) = \exp[(g/c_s^2)z], \qquad c = c_s. \tag{146}$$

The relevant motion is not irrotational. Isobars are not streamlines in a frame of reference moving with the waves. (The opposite statement in Yih [1960a] is incorrect.) The curious thing about this solution is that the wave motion progresses with sound speed for all values of the wavelength but is not longitudinal.

The solution (146) is still good for three-dimensional flows, except that $c = k'c_s/k$, with k' defined in (145).

From the mathematical point of view the case of infinite depth is interesting because the spectrum of the eigenvalues may be continuous, aside from isolated eigenvalues, some or all of which are associated with the free surface and the interfaces. For example consider the case of an incompressible fluid of infinite depth, with a density given by

$$\bar{\rho} = \rho_0 \, e^{-bz},$$

with z extending from 0 to $-\infty$. The solution of (31) is

$$w = Ae^{a_1 z} + Be^{a_2 z},$$

in which

$$(a_1, a_2) = \frac{1}{2}\left\{b \pm \left[b^2 - 4k^2\left(\frac{gb}{\sigma^2} - 1\right)\right]^{1/2}\right\}, \qquad \sigma = kc.$$

If $gb \leq \sigma^2$, a_1 is positive and a_2 negative, and to satisfy the condition $w(-\infty) = 0$, B must be put equal to zero. The surface condition (123) then demands that

$$a_1 = \frac{gk^2}{\sigma^2} = \frac{g}{c^2}, \qquad \text{or} \qquad c^2 = \frac{g}{k}. \tag{147}$$

The solution is therefore the irrotational one discussed earlier in this section. However, if $gb > \sigma^2$, either both a_1 and a_2 are positive, or else they both have positive real parts. The condition at $z = -\infty$ is therefore automatically satisfied. The surface condition demands

$$Aa_1 + Ba_2 - (A + B)gk^2/\sigma^2 = 0. \tag{148}$$

Thus given b, k, σ, and B, A can be solved. Thus any σ less than $\sqrt{gb}$ will do, and the spectrum for c is

$$0 < c < \frac{\sqrt{gb}}{k}, \tag{149}$$

and is continuous. Taking $B = 0$, we recover the solution (147), which is therefore not restricted to the case $gb \leq \sigma^2$.

The free-surface waves characterized by $\sigma^2 = gk$ will make gb/σ^2 greater than 1 if $k < b$. In that case the eigenvalue for the free-surface mode is embedded in the continuous spectrum. If $k \geq b$, gb/σ^2 (for $\sigma^2 = gk$) is less or equal to 1, and the eigenvalue for the free-surface mode is isolated. The fact that for $k < b$ the speed of free-surface waves is imbedded in a continuous spectrum of c indicates that free-surface waves are not necessarily faster than internal waves.

If the upper surface is rigid, no solution is possible for $gb \leq \sigma^2$. This means that no free-surface waves are possible, as expected. The continuous spectrum

$$\sigma^2 < gb$$

all corresponds to internal waves. In this connection it should be noted that even if $k < b$, so that the value $\sigma^2 = gk$ for irrotational free-surface waves is less than gb, such waves are not possible for a rigid upper boundary. The mode corresponding to a rigid upper boundary and to $\sigma^2 = gk$ in the continuous spectrum is always distinct from the free surface mode in that for the former $B = A \neq 0$, whereas for the latter, as we have seen, $B = 0$.

Associated with (144) is the edge-wave solution

$$\begin{aligned} -iu &= \frac{k}{k^2 - l^2} w', \qquad ilu = kv, \\ w &= A \exp\left[ik(x - ct) - ly + (k^2 - l^2)^{1/2} z\right], \\ c^2 &= \frac{g(k^2 - l^2)^{1/2}}{k^2}, \end{aligned} \tag{150}$$

which is the same as Stokes' solution for a homogeneous fluid. It is periodic and propagates in the x-direction and attenuates with increasing y, the distance from the shore. The first two equations in (149) follow from (24) and (27), on changing l to il and setting c_s^2 equal to infinity. It can be verified that the free-surface condition is satisfied, and that in a direction normal to the sloping bank $z = -y \tan\beta$ the velocity is zero, where

$$\tan\beta = \frac{\sqrt{k^2 - l^2}}{l}. \tag{151}$$

Associated with (146) is the edge-wave solution

$$\begin{aligned} -iu &= \frac{k}{k^2 - l^2} w', \qquad ilu = kv, \\ w &= A \exp\left[ik(x - ct) - ly + \frac{g}{c_s^2} z\right], \qquad c = \frac{(k^2 - l^2)^{1/2} c_s}{k}. \end{aligned} \tag{152}$$

Again, the velocity component normal to the sloping bank $z = -y \tan\beta$, with β defined above, is zero.

13. PROPAGATION OF DISTURBANCE IN THREE DIMENSIONS — GROUP VELOCITY

Although normal modes of three-dimensional waves propagating in two horizontal dimensions can be obtained from normal modes of two-dimensional waves propagating in one dimension, as has been shown in Section

4, knowledge concerning such waves does not illuminate the propagation of disturbances in three dimensions, which will be briefly discussed here.

For the sake of simplicity, we shall consider an incompressible fluid with the mean-density distribution

$$\bar{\rho} = \bar{\rho}_0 e^{-2\beta z}.$$

Then N^2 is just $2\beta g$, since that is the constant value of $-g\bar{\rho}'/\bar{\rho}$. Suppose that the fluid is set in motion by a body oscillating with frequency σ. Then the governing equation (30) can be written, with σ replacing kc,* k_x replacing k, and k_y replacing l, as

$$(N^2 - \sigma^2)k_r^2 w + \sigma^2(w'' - 2\beta w') = 0, \tag{153}$$

the solution of which is

$$w(z) = e^{\beta z}(Ae^{ik_z z} + Be^{-ik_z z}), \tag{154a}$$

$$k_z = \left[\left(\frac{N^2}{\sigma^2} - 1\right)k_r^2 - \beta^2\right]^{1/2}, \qquad k_r^2 = k_x^2 + k_y^2. \tag{154b}$$

The solution for the vertical velocity component w is the real part of the product of the right-hand side of (154a) and the exponential factor $\exp[i(k_x x + k_y y - \sigma t)]$. From (154b) it can be seen that k_z^2 is positive or negative accordingly as

$$\sigma \lessgtr \sigma_c = \frac{N}{\sqrt{1 + (\beta/k_r)^2}}$$

Equation (154b) can be rewritten as

$$\frac{\sigma}{N} = \frac{k_r}{(k_r^2 + k_z^2 + \beta^2)^{1/2}}. \tag{154c}$$

If k_z^2 is positive, so that the motion is oscillatory in all three directions, the phase velocity is defined by the kinematic relation

$$\mathbf{c} = \frac{\sigma}{k}\mathbf{n}, \tag{155}$$

in which

$$k = |\mathbf{k}|, \qquad \mathbf{k} = (k_x, k_y, k_z), \qquad \text{and} \qquad \mathbf{n} = \mathbf{k}/k. \tag{156}$$

Then†

$$c = |\mathbf{c}| = \frac{\sigma}{k} = \frac{Nk_r}{k(k^2 + \beta^2)^{1/2}}. \tag{157}$$

* See the footnote following Eq. (23).

† Note that this definition of c is not the same as the definition of c as σ/k_x, which is the velocity of wave propagation in the x-direction.

Equations (154c) and (157) show that for a given k waves propagate the fastest in the horizontal direction. Note that if *real* k_x and k_y are prescribed, and the fluid entends from $z = 0$ to $z = d$, then the boundary conditions determine the eigenvalues of k_z and hence of σ. Then the mode is the normal mode of three-dimensional waves propagating in two dimensions. However if we regard the disturbance as arbitrary and σ as given, (154c) merely gives a relationship between k_z and k_r, and the solution (154a) does not represent a normal mode propagating in two dimensions but stands for a rather more general disturbance, which is dynamically possible.

The group velocity $\mathbf{c}_g$, defined by

$$\mathbf{c}_g = \left(\frac{\partial\sigma}{\partial k_x}, \frac{\partial\sigma}{\partial k_y}, \frac{\partial\sigma}{\partial k_z}\right), \tag{158}$$

is given by

$$(c_{gx}, c_{gy}) = \frac{A}{k_r}\frac{k_z^2 + \beta^2}{k^2 + \beta^2}(k_x, k_y), \qquad c_{gz} = -\frac{Ak_r k_z}{k^2 + \beta^2}. \tag{158a}$$

Wu (1967) noted that $c_{gy}/c_{gx} = c_y/c_x$, and hence $\mathbf{c}$ and $\mathbf{c}_g$, are in the same vertical plane. For a disturbance with a specified σ, (154c) becomes

$$\left(\frac{\beta}{k}\right)^2 = M^2 \sin^2\theta - \cos^2\theta, \tag{159}$$

where

$$M^2 = \left(\frac{N}{\sigma}\right)^2 - 1, \qquad \theta = \tan^{-1}(k_r/k_z).$$

Then

$$\theta_M \le \theta \le \pi - \theta_M, \qquad \theta_M = \cot^{-1} M, \tag{160}$$

which was found by Görtler (1943) by consideration of the characteristics. Wu (1967) noted that, from (158) and (154c),

$$\frac{c_{gz}}{c_{gr}} = -\frac{k_r k_z}{k_z^2 + \beta^2} = -\frac{1}{M^2}\cot\theta,$$

so that disturbances are bounded away from the vertical cone $r = \pm Mz$, and can only reach the region outside of it, confirming Görtler's result by group-velocity arguments.

Mowbray and Rarity (1967) investigated the propagation in the x-z plane, and confirmed (160), for $k_y = 0$, both analytically and experimentally. Experimental verification of (160) for wave motion in the x-z plane, however, was carried out first by Görtler (1943).

Mowbray and Rarity (1967) discussed the case of short waves, for which the $k_z^2 + \beta^2$ in (158a) can be replaced by k_z^2, and found the interesting result that for such waves **c** is perpendicular to $\mathbf{c}_g$. Since **c** is normal to the crests of the waves, this means that $\mathbf{c}_g$ is along them, a most intriguing situation. Energy is thus propagated from the source of disturbance along the crest of waves in the permissible region specified by (160). If the disturbance is created by a vertical plate oscillating vertically, the sources of disturbance are the edges of the plate.

The group velocity defined by (158) has kinematic and dynamic significances. After the waves of all wave numbers have sufficiently dispersed, so that

$$\frac{\partial \mathbf{k}}{\partial t} + \mathbf{grad}\, \sigma = 0 \tag{161}$$

is valid, it can be shown that $\mathbf{c}_g$ defined by (158) is the velocity with which wave numbers are propagated, or the velocity with which a point with a given **k** moves in space, provided N is constant. If N varies σ is a function not only of **k** but also of z, and the wave number is no longer conserved on a point moving with the group velocity. Waves are then reflected internally and may well be trapped in regions of limited vertical extent, as shown by Mowbray and Rarity (1967). From a dynamical point of view, if N is constant, as is the case here, the demonstration of Broer (1950) by the method of stationary phase applies, and the group velocity is the velocity of energy propagation. It should however be remembered that here also it is tacitly assumed that the different wave components have been sufficiently dispersed for (161) to be valid. Otherwise there would arise the question of for what wave number the group velocity is evaluated. Also, since the method of stationary phase is used, the shorter the waves the more exact the conclusion of Broer.

A few words are necessary to explain why, for the normal modes of two-dimensional waves propagating horizontally, the group velocity is in the same direction as the phase velocity, whereas for propagation in the x-z plane this is not so, and for $\beta^2 \ll k_z^2$ the two velocities can even be perpendicular. The former is in fact the result of superposition of infinitely many disturbances continually reflected at both the upper and lower boundaries. Although the energy of each reflected disturbance is propagated along the crests, the resultant propagation of the interwoven disturbances is in the x-direction.

It is not inappropriate to mention here the study of Wu and Mei (1967) of the two-dimensional waves created in a stratified fluid by a line source moving horizontally in a direction normal to its length. The two-dimensional motion due to a line disturbance moving arbitrarily has been studied by Rarity (1967), and the three-dimensional motion produced by a point disturbance moving arbitrarily has been studied by Wu (1967).

14. APPROXIMATE CALCULATION OF EIGENVALUES

In principle, the eigenvalues of the phase velocity c for wave motion can be obtained by solving the differential equation numerically or analytically, arranging to have the lower boundary condition $w(0) = 0$ satisfied and using the interfacial conditions (if there are interfaces), and finally solving for c^2 after the boundary condition at $z = d$ is imposed. With an electronic computer it is probably more convenient to assume values of c^2 to start with and to verify their suitability as eigenvalues by the upper boundary condition. In practice, such a calculation, by hand or by machine, is too laborious and should be avoided unless a high degree of accuracy is justified. Two methods for calculating the eigenvalues approximately will be presented in this section. The first is a method suitable for small density stratification in each continuously stratified layer. The second method is suitable if the density distribution in each continuously stratified layer, not necessarily small, is nearly exponential. All these methods are applicable only if the differential equation is of the type (31), (66a), (66b), or (66c).

We note, first of all, that for internal waves of an incompressible fluid governed by the differential equation (31) and the boundary conditions (33), there is an upper bound for the frequency σ. Writing $c = \sigma/k$ in (31), one sees immediately, with the aid of Sturm's first comparison theorem given in Section 8, or by multiplying (31) by w and integrating between 0 and the depth d, that if the first condition in (33) is satisfied the second condition in (33) cannot be satisfied if

$$\sigma^2 \geq \max\left(-\frac{g\bar{\rho}'}{\bar{\rho}}\right) \equiv N^2.$$

Hence for truly internal wave motion

$$\sigma^2 < N^2. \tag{162}$$

The N defined above is called the Brunt–Väisälä frequency. It is not an attainable frequency for truly internal waves but an upper bound for it. It should also be borne in mind that N exists only if there is no free surface and no internal density discontinuities.

14.1. *Eigenvalues for Interfacial Waves and for Slight Continuous Stratification in Each Layer*

Since the eigenvalues of c^2 for interfacial waves are largely independent of the continuous stratification in each layer if that stratification is slight, the calculation of such eigenvalues can be considerably simplified by a perturbation procedure. Let the density in one particular layer be

$$\bar{\rho} = \bar{\rho}_0 + \bar{\rho}_1(\eta), \tag{163}$$

in which $\bar{\rho}_0$ is constant and $\bar{\rho}_1$ is everywhere small compared with $\bar{\rho}_0$. Let

$$w = w_0 + w_1, \qquad w_0 = A \sinh m\eta + B \cosh m\eta, \tag{164}$$

in which $w_1(\eta)$ is everywhere small compared with 1. Substitution of (163) and (164) into (31), with m replacing k and λ replacing g/c^2, gives

$$w_1'' - m^2 w_1 = -\frac{\bar{\rho}_1'}{\bar{\rho}_0}(w_0' - \lambda w_0), \tag{165}$$

if second-order terms in $\bar{\rho}_1$ or w_1 are neglected. All that is needed is to find a particular solution of w_1 in (165). This is easily done by assuming

$$w_1 = w_2 e^{m\eta},$$

which immediately gives

$$(w_2' e^{2m\eta})' = -\frac{\bar{\rho}_1'}{\bar{\rho}_0}(w_0' - \lambda w_0)e^{m\eta}.$$

Two integrations, with constants of integration entirely neglected, give

$$w_2 = -\frac{1}{\rho_0}\int \left[e^{-2m\eta}\int \bar{\rho}_1'(w_0' - \lambda w_0)e^{m\eta}\, d\eta\right] d\eta.$$

The w_2, w_1, and w so obtained are all of the form

$$A f(\eta) + B\, g(\eta),$$

in which f and g are linear functions of λ. After the forms of the eigenfunctions for all the layers have been determined, the boundary conditions can be applied, and the eigenvalues found. If the number of layers is less than 4, the calculation for λ is straightforward and simple.

The method presented in the preceding paragraph is for interfacial waves. For waves internal to at least one layer approximate values of the eigenvalues can be found by treating the interfaces as rigid, provided the speeds of these waves are much smaller than those of the interfacial waves, that is, provided $d_i\bar{\rho}_i'$ ($i = 1, 2$, etc.) is much less than $\Delta\rho_j$ ($j = 1, 2$, etc.) at any location, and for any i and any j. This provision is in fact implied in the statement that the continuous stratification is slight.

14.2. *Eigenvalues for Superposed Layers with Nearly Exponential Stratification*

If the density in each layer is nearly exponentially stratified, so that

$$\bar{\rho} = \bar{\rho}_0 + \bar{\rho}_1(\eta), \qquad \bar{\rho}_0 = a e^{-\beta\eta},$$

in which a is constant and $\bar{\rho}_1$ is small compared with $\bar{\rho}_0$ everywhere, the eigenfunction for each layer is of the form

$$w = w_0 + w_1, \qquad w_0 = Ae^{\alpha_1\eta} + Be^{\alpha_2\eta},$$

with α_1 and α_2 given in the second equation after (159). The resulting equation is

$$(\bar{\rho}_0 w_1')' - m^2 \bar{\rho}_0 w_1 - \lambda \bar{\rho}_0' w_1 = -(\beta \bar{\rho}_1 + \bar{\rho}_1')(w_0' - \lambda w_0).$$

With

$$w_1 = e^{\alpha_1 \eta} w_2,$$

the differential equation becomes

$$(\bar{\rho}_0 e^{2\alpha_1 \eta} w_2')' = -e^{\alpha_1 \eta}(\beta \bar{\rho}_1 + \bar{\rho}_1')(w_0' - \lambda w_0),$$

from which the particular solution is found after two quadratures. The eigenvalues are then found by imposing the boundary conditions. All the eigenvalues, except possibly those of very high modes (for which c^2 is very small and λ very large), can be found by this method.

All or some of the methods given in this entire section (Section 14) are applicable to compressible fluids provided (66) can be simplified to (66a), (66b), or (66c)—that is, provided the waves are predominantly gravity waves or predominantly acoustic waves. The method given in Sections 14.1 and 14.2 is a perturbation method which can be applied whenever an exact solution is known for a density stratification near the actual one. The constant-density case and the case of exponential density distribution are merely special cases for which the exact solutions are known. In fact, if

$$w_0 = A\,f(\eta) + B\,g(\eta) \qquad \text{and} \qquad w_1 = f(\eta)\,w_2,$$

the particular solution for w_2 is to be found from

$$\{\bar{\rho}_0(\eta)\,[f(\eta)]^2 w_2'\}' = -f(\eta)\,\bar{\rho}_1\left(\frac{\bar{\rho}_1'}{\bar{\rho}_1} - \frac{\bar{\rho}_0'}{\bar{\rho}_0}\right)(w_0' - \lambda w_0),$$

by two quadratures.

15. WAVES GENERATED BY A PLANE WAVE MAKER

So far eigenvalues for c^2 for a fixed wave number k have been considered. There are problems in which the frequency σ is prescribed. The problem of finding the waves generated by a wave maker is one of them. Since the problem of waves generated in an incompressible fluid by a plane wave maker is important for experimental studies of such waves, it will be discussed in detail.

Let the oscillating plane be situated at $x = 0$ and oscillate as $a \cos \sigma t$, and let the fluid extend from there to $x = \infty$. The bottom corresponds to $z = 0$, and the free surface is at $z = d$. For an incompressible fluid the continuity equation is (1.10), and the velocity components $u(x, z)$ and $w(x, z)$ in two-dimensional flows can be expressed in terms of the stream function ψ:

$$u(x, z) = \frac{\partial \psi}{\partial z}, \qquad w(x, z) = -\frac{\partial \psi}{\partial x}. \tag{166}$$

Furthermore, since the fluid is incompressible, the linearized equation of incompressibility is, from (1.9) on setting $v = u_2 = 0$,

$$\frac{\partial \rho}{\partial t} + w\bar{\rho}' = 0. \tag{167}$$

From the first and third equations of motion in (8), and from (166) and (167), it follows by elimination of p that

$$(\sigma^2 \bar{\rho} + g\bar{\rho}')\frac{\partial^2 \psi}{\partial x^2} + \sigma^2 \frac{\partial}{\partial z}\left(\bar{\rho}\frac{\partial \psi}{\partial z}\right) = 0. \tag{168}$$

With

$$\psi = f(z) \exp i(kx - \sigma t), \tag{169}$$

(168) becomes

$$(\bar{\rho}f')' - k^2\left(\bar{\rho} + \frac{g\bar{\rho}'}{\sigma^2}\right)f = 0, \tag{170}$$

which is the same as (31), as it should be, since $w(z) = -ikf(z)$ according to (166) and (169). For a given σ, there are infinitely many real eigenvalues for k^2, some of which positive, some of which negative. For $k = 0$, w is identically zero, and if ψ were not zero there would be, at all values of t except those for which $\sin \sigma t = 0$, a constant (with respect to x) pressure gradient in the x-direction throughout the fluid—a situation which the free surface will not allow. Hence the solution is a superposition of the real parts of elementary solutions of the type (169), with $k \neq 0$. With the eigenvalues of k denoted by k_n, it is

$$\frac{\psi}{a^2\sigma} = \sum_{n=-\infty}^{n_1-1} A_n f_n(z) \exp(-|k_n|x) \cos \sigma t + \sum_{n=n_1}^{\infty} A_n f_n(z) \cos(k_n x - \sigma t),$$

in which the f's are the eigenfunctions, and the A's are to be determined by the condition at the wave maker ($x = 0$):

$$z = \sum_{-\infty}^{\infty} A_n f_n(z).$$

If the upper surface is rigid, the solution is

$$\frac{\psi}{a^2\sigma} = \cos \sigma t \left(\int_0^d \frac{dz}{\bar{\rho}}\right)^{-1} \int_0^z \frac{dz}{\bar{\rho}} + \sum_{n=1}^{n_1-1} A_n f_n(z) \exp(-|k_n|x) \cos \sigma t$$
$$+ \sum_{n=n_1}^{\infty} A_n f_n(z) \cos(k_n x - \sigma t),$$

in which the first term corresponds to the solution of (170) with $k = 0$, and its coefficient is determined by the fact that $\psi/a^2\sigma$ is equal to $\cos \sigma t$ at $z = d$, and $f_n(d) = 0$ for a rigid upper boundary. The coefficients A_n are to be determined from

$$z = \left(\int_0^d \frac{dz}{\bar{\rho}}\right)^{-1} \int_0^z \frac{dz}{\bar{\rho}} + \sum_{n=-\infty}^{\infty} A_n f_n(z).$$

The eigenfunctions $f_n(z)$ are complete in either case.

With

$$\xi = \int_0^z \frac{dz}{\bar{\rho}},$$

(168) may be written as

$$(\sigma^2 \bar{\rho}^2 + g\bar{\rho}\bar{\rho}') \frac{\partial^2 \psi}{\partial x^2} + \sigma^2 \frac{\partial^2 \psi}{\partial \xi^2} = 0, \qquad (171)$$

which (Görtler, 1954) is of the elliptic or hyperbolic type according as $\sigma^2 \bar{\rho} + g\bar{\rho}'$ is positive or negative. If the type is elliptic, the corresponding disturbance is local. If the type is hyperbolic, the corresponding disturbance is wavelike and real characteristics exist, along which discontinuities will propagate.

If the frequency of wave motion is very low, so that σ is very small, the characteristics are nearly horizontal. This situation is intimately related to the trainlike behavior of a stratified fluid described in Section 3 of Chapter 1.

16. ATMOSPHERIC WAVES IN THE LEE OF MOUNTAINS

Atmospheric observations have indicated the existence of alternately ascending and descending air currents above a mountain range, and stationary waves at various heights on the lee side of it, called lee waves in the literature. Since the atmosphere is often stratified in entropy (or potential density or temperature), the existence of these waves is perhaps not surprising. They are, in fact, entirely of the same nature as the surface waves in a stream of water on the lee side of a barrier, studies of which can be found in the literature of classical hydrodynamics. [See, for instance, Lamb 1945, pp. 409–416.] Nevertheless the study of lee waves in the atmosphere has had a history dating only from the early 1940s. As is characteristic of a subject of relatively recent interest, its development is not entirely free from controversies.

The motion under study here differs from the wave motion governed by (28) in two respects. It is steady with respect to the solid boundary (the ground), and it has a wind profile not necessarily uniform. Only the perturbation theory for two-dimensional flow will be presented. Lee waves of arbitrary amplitude will be discussed in Chapter 3, Section 7. The few papers on three-dimensional lee waves are all based on perturbation theory. References to them will be made at the end of this section.

The linearized equations of motion are, if viscosity is neglected,

$$\bar{\rho}Uu_x + \bar{\rho}U'w = -p_x, \tag{172}$$

$$\bar{\rho}Uw_x = -p_z - g\rho, \tag{173}$$

in which $\bar{\rho}$ and U are the density and the velocity of the undisturbed fluid, respectively, subscripts indicate partial differentiation, and the accent indicates differentiation with respect to the vertical coordinate z. The other Cartesian coordinate, x, is measured along the direction of U, assumed horizontal. Both $\bar{\rho}$ and U are functions of z only. The quantities u and w are velocity components in the directions of x and z for the perturbation flow, respectively, and p and ρ are the perturbation pressure and density. The pressure $\bar{p}$ of the undisturbed fluid is related to its density $\bar{\rho}$ by the hydrostatic condition (4). The equation of continuity is, in its linearized form,

$$\bar{\rho}(u_x + w_z) + U\rho_x + \bar{\rho}'w = 0. \tag{174}$$

If heat conduction as well as viscosity is neglected, the entropy of a fluid particle will remain unchanged as it moves about, or

$$\frac{D}{Dt}\left[\frac{\bar{p} + p}{(\bar{\rho} + \rho)^\gamma}\right] = 0, \tag{175}$$

in which γ is defined as in (1.5) and

$$\frac{D}{Dt} = U\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}.$$

The linearized form of (175) is

$$Up_x + w\bar{p}' - c_s^2(U\rho_x + w\bar{\rho}') = 0, \tag{176}$$

in which c_s is the speed of sound defined as in (6) and is a function of z only.

The quantities p, ρ, and u can be eliminated from (172), (173), (174), and (176), to yield an equation in w alone, which is the equation governing lee-wave motion in the atmosphere. Since this equation is basic to lee-wave theories, its derivation will be presented. Elimination of ρ from (174) and (176) yields

$$c_s^2\bar{\rho}(u_x + w_z) + Up_x + w\bar{p}' = 0. \tag{177}$$

Elimination of p_x from (172) and (177) then gives

$$\bar{\rho}(c_s^2 - U^2)u_x = -[c_s^2\bar{\rho}w_z + (\bar{p}' - \bar{\rho}UU')w], \tag{178}$$

which relates u to w. If, in the equation resulting from the elimination of p from (172) and (173), ρ_x is replaced by its equivalent obtained from (174), another relationship between u and w is obtained:

$$(\bar{\rho}Uu_x)_z + \frac{g}{U}\bar{\rho}u_x = \bar{\rho}u_x = \bar{\rho}Uw_{xx} - \frac{g}{U}(\bar{\rho}w)_z - (\bar{\rho}U'w)_z. \tag{179}$$

Finally, substitution of (178) into (179) yields the equation sought, after $\bar{p}'$ is replaced by $-g\bar{\rho}$:

$$\left\{\frac{U\bar{\rho}}{c_s^2 - U^2}[c_s^2 w_z - (UU' + g)w]\right\}_z + \frac{g\bar{\rho}}{U(c_s^2 - U^2)}[c_s^2 w_z - (UU' + g)w] + \bar{\rho}Uw_{xx} - \frac{g}{U}(\bar{\rho}w)_z - (\bar{\rho}U'w)_z = 0. \qquad (180)$$

In the atmosphere it can be assumed that $c_s^2 \gg U^2$ and $g \gg UU'$, and (180) can be replaced without serious error by

$$\nabla^2 w - \beta w_z + \left(\frac{g\beta_1}{U^2} - \frac{U''}{U}\right)w = 0, \qquad (181)$$

in which

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \qquad \beta = -\frac{\bar{\rho}'}{\bar{\rho}}, \qquad \text{and} \qquad \beta_1 = \beta - \frac{g}{c_s^2}. \qquad (182)$$

Equation (181) was obtained by Scorer [1949], among others. The two terms in the parenthesis of (181) were shown by him to be of the same order of magnitude in the atmosphere. In Scorer's derivation the potential density is used, which made the derivation a little simpler. It should be noted that the β in Scorer's paper and in the paper of Lyra [1943] is not the β in Eq. (181), but is the β_1 here, which is equal to T_p'/T_p, T_p being the potential temperature, or to S'/S, S being the entropy.

Queney [1941, 1948] considered a constant U and an isothermal atmosphere. In that case the equation of state and (4) give

$$\bar{\rho} = \rho_0 \exp\left(-\frac{\gamma g z}{c_s^2}\right) = \rho_0 \exp\left(-\frac{gz}{RT_0}\right), \qquad (183)$$

so that β is equal to $\gamma g/c_s^2$ and $\beta_1 = (\gamma - 1)g/c_s^2$. Thus β and β_1 are of the order of g/c_s^2. This is generally true for the atmosphere. Since U is constant, (181) can be written as

$$\nabla^2 w - \beta w_z + \frac{g\beta_1}{U^2} w = 0. \qquad (184)$$

Queney considered a ground shape

$$\zeta_0 = \frac{b}{1 + (x/a)^2}. \qquad (185)$$

If ζ is the displacement of a streamline from its elevation far upstream,

$$\frac{D\zeta}{Dt} = w,$$

which, in the linearized form, is

$$U\zeta_x = w, \qquad (186)$$

since the flow is steady. If w and ζ are assumed to have the form

$$w = w(z)\, e^{ikx}, \qquad \zeta = \zeta(z)\, e^{ikx}, \tag{187}$$

then

$$ik\, U\zeta(z) = w(z), \tag{188}$$

and (184) can be written

$$\zeta''(z) - \beta\, \zeta'(z) + \left(\frac{g\beta_1}{U^2} - k^2\right) \zeta(z) = 0, \tag{189}$$

the solution of which is straightforward since the coefficients are constant. Multiplication of the fundamental solution by the function $f(k)e^{ikx}$ and summing over k produces the solution (real part to be used)

$$\begin{aligned} \zeta &= \exp\left(\frac{gz}{2RT_0}\right) \cdot \int_0^\infty f(k) \exp\left(ikx - \sqrt{k^2 - k_s^2}\, z\right) dk \\ &= \sqrt{\frac{\rho_0}{\bar{\rho}}} \int_0^\infty f(k) \exp\left(ikx - \sqrt{k^2 - k_s^2}\, z\right) dk, \end{aligned} \tag{190}$$

with

$$k_s^2 = \frac{g\beta_1}{U^2}, \tag{191}$$

from which a term $-\beta^2/4$ has been neglected. Applying (190) at $z = 0$ (rather than at $z = \zeta$, as a nonlinear theory would demand), we have

$$\zeta_0 = \int_0^\infty f(k) \cos kx\, dk.$$

Since

$$ab \int_0^\infty \cos kx \exp(-ka)\, dk = \frac{b}{1 + (x/a)^2}, \tag{192}$$

it follows that

$$f(k) = ab \exp(-ka),$$

and

$$\zeta = ab\sqrt{\frac{\rho_0}{\bar{\rho}}} \int_0^\infty \exp\left(ikx - ka - \sqrt{k^2 - k_s^2}\, z\right) dk. \tag{193}$$

The flow pattern corresponding to (193) is strongly dependent on the magnitude of ak_s. If $k_s = 0$, which means that there is no or very weak stratification, (193) becomes

$$\begin{aligned} \zeta &= ab\sqrt{\frac{\rho_0}{\bar{\rho}}} \int_0^\infty \cos kx \exp(-ka - kz)\, dk \\ &= \sqrt{\frac{\rho_0}{\bar{\rho}}}\, \frac{b}{1 + [x/(a + z)]^2}, \end{aligned} \tag{194}$$

so that there are no waves created by the mountain. If k_s^2 is greater than zero, however slightly, there will always be lee waves of the gravity type, since an isothermal atmosphere is infinite in extent, and there are always waves which have a speed equal to U, provided β_1 is not zero. These waves can therefore be stationary. However, if an upper boundary, fixed or free, is supposed to exist at $z = d$, there is a maximum speed (see Sections 9 and 10), and a minimum value of $g\beta_1 d^2/c^2$ for gravity waves. If

$$(k_s d)^2 = \frac{g\beta_1 d^2}{U^2}$$

is less than this minimum, gravity waves cannot stay stationary and lee waves of the gravity type cannot exist. This perhaps implies that, for an isothermal atmosphere of infinite extent, if $k_s a$ is small, lee waves of the gravity type, though they can exist, are rather weak. Further clarification of this point is desirable.

For $ak_s = 1$, the flow pattern is given in Fig. 6, after Queney [1948]. As a becomes larger and larger, lee waves having their origin in Coriolis acceleration will appear. See Queney [1947, 1948] and Scorer [1949]. Only lee waves of the gravity type are discussed here.

The value of $(k^2 - k_s^2)^{1/2}$ in (193) is positive if $k > k_s$. This requirement is necessary if $\zeta(z)$ is to remain finite as $z \to \infty$. For $k_s > k$, Queney, Lyra (1943), and most later workers in this field took the radical to be

$$-i(k_s^2 - k^2)^{1/2}.$$

This amounts to taking the contour in the k-plane *below* (Fig. 7) the branch point $k = k_s$. Scorer [1949, 1953, and 1954] took it to be

$$+i(k_s^2 - k^2)^{1/2},$$

which implies that the contour is taken above the branch point. The controversy went on for a decade, until Crapper [1959] considered the establishment of flow and showed that the branch point is, before steady flow is established, *above* the real axis in the complex k-plane. Thus the contour is below the branch point before the limit is reached, and remains below in the limit. The mathematical formalism is the same if Rayleigh's device of introducing a small frictional force proportional to the velocity is used. Crapper's demonstration is a straightforward extension of the one for water waves in the lee of a barrier. See, for instance, p. 401 of Lamb [1945]. The question of the proper contour to take has been called the question of the "upper boundary condition." The numerous discussions of this question in the literature seem to have had the unintended effect of exaggerating the importance of Scorer's oversight and blurring his valuable contributions.

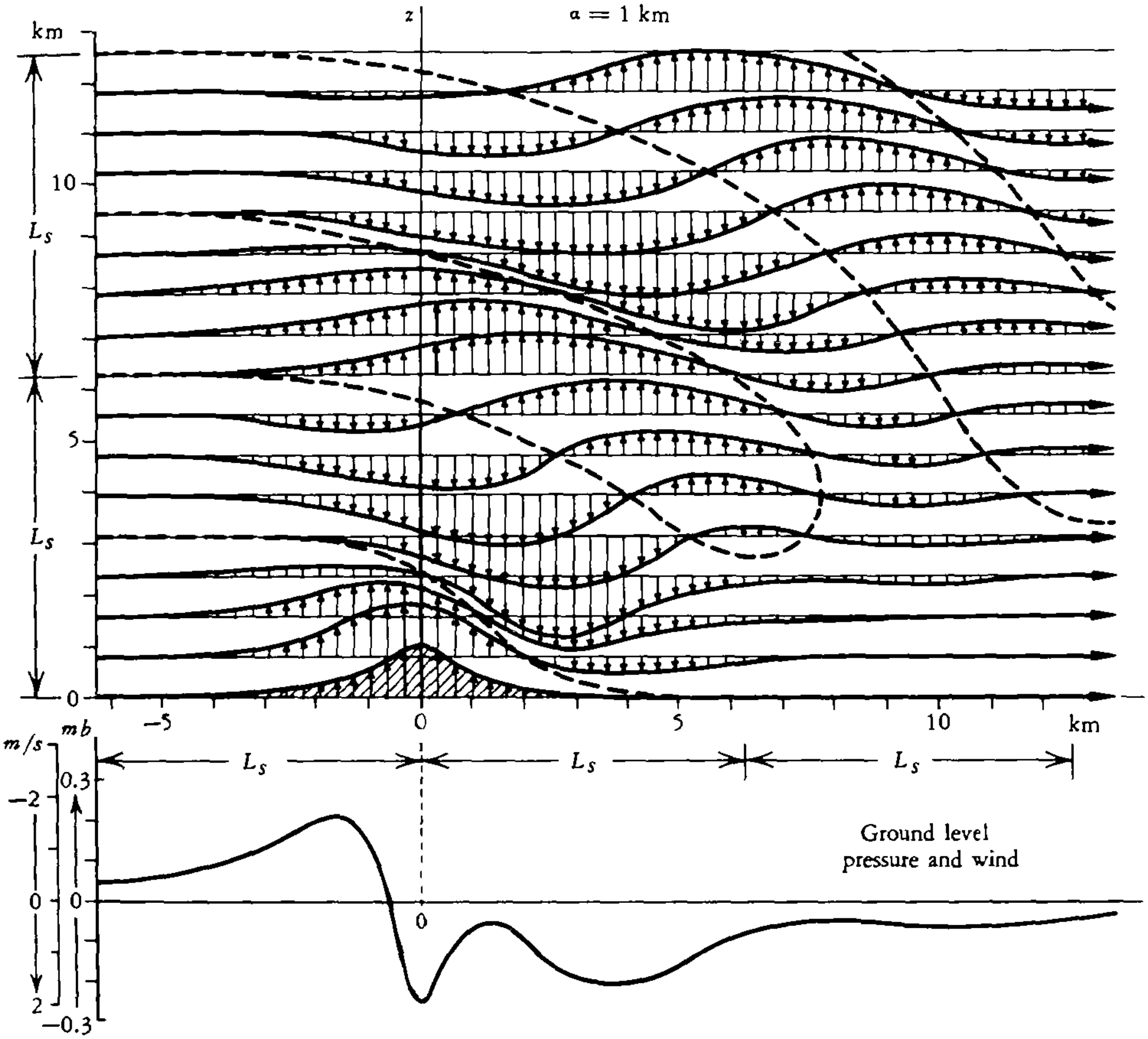


FIGURE 6. Lee-wave pattern for an unbounded isothermal atmosphere for $ak_s = 1$ and $a = 1$ km, after Queney [1948]. (*Courtesy of the American Meteorological Society.*)

Lyra [1943] considered lee waves in an atmosphere with a temperature decreasing linearly with respect to z. It can be shown that this is a polytropic atmosphere with ($R = \bar{p}/\bar{\rho}\bar{T}, \gamma = -d\bar{T}/dz$)

$$\frac{\bar{p}}{\bar{p}_0} = \left(\frac{\bar{\rho}}{\bar{\rho}_0}\right)^n, \qquad n = \frac{g}{g - R\gamma}.$$

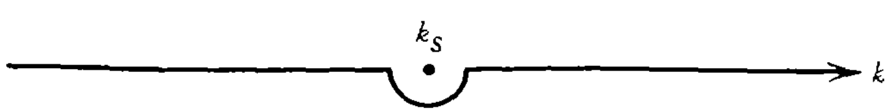


FIGURE 6a. Contour for determining the sign of $(k^2 - k_s^2)^{1/2}$.

The governing differential equation has constant coefficients if $\gamma = 0$. In this regard Lyra's problem resembles Queney's. Lyra's solution is, however, quite different in approach. A Green's function constructed from the solution of the differential equation is used to obtain the solution for any arbitrary ground from $\zeta_0 = \zeta(x, 0)$. In the construction of Green's function, Lyra took measures to ensure that there are no waves far upstream. The solution is quite general as far as ground form is concerned, and constitutes a significant contribution to the field. Wind shear is, however, not considered.

Scorer [1949] generalized the results of Queney and Lyra by considering an atmosphere of two layers, in each of which the quantity

$$l^2 = \frac{g\beta_1}{U^2} - \frac{U''}{U} \tag{195}$$

is constant. The upper layer may be regarded as the stratosphere and the lower one the troposphere, although this identification is not necessary. The equation to be solved is (181), which, with (187), now assumes the form

$$w'' - \beta w' + (l^2 - k^2)w = 0,$$

which has the solutions $e^{\lambda_1 z}$ and $e^{\lambda_2 z}$, with

$$(\lambda_1, \lambda_2) = \tfrac{1}{2}(\beta \pm \mu), \qquad \mu^2 = \beta^2 + 4(k^2 - l^2). \tag{196}$$

Scorer neglected the term $\beta/2$ in λ_1 and λ_2 and the term β^2 in μ, and demanded that the fundamental solution satisfy

(i) continuity of w at the interface,
(ii) continuity of u at the interface,
(iii) $w = 0$ at $z = -h$.

In (iii), z is measured upward from the interface of the two layers, and h is the thickness of the lower layer, so that (iii) demands w to be zero on the flat ground $z = -h$. While this is not necessary in Queney and Lyra's problem it is quite necessary for Scorer's, because discrete lee-wave modes exist in Scorer's model, which will not damp out as $x \to \infty$. If (iii) is not satisfied the ground will not be level at large (positive) values of x downstream from the mountain. Condition (ii) is valid because there is no discontinuity either in $\bar{\rho}$ or in U, so that no vortex sheet can be produced in the fluid. Condition (i) is valid because U is continuous.

The fundamental solution is, with the neglect of $\beta/2$ in λ_1 and λ_2 and of β^2 in μ,

$$\begin{aligned} w_2 &= Ae^{-\mu_2 z}, \\ w_1 &= A\left(\cosh \mu_1 z - \frac{\mu_2}{\mu_1} \sinh \mu_1 z\right), \end{aligned} \tag{197}$$

in which the subscripts 1 and 2 denote the lower and upper layers, respectively. Thus, μ_1 corresponds to l_1 and μ_2 to l_2. Equations (197) satisfy the interfacial conditions (i) and (ii). The first of (197) satisfies the condition of boundedness of w_2 at $z = \infty$ if the factor $\exp(-\beta z/2)$ is ignored, as by Scorer, and in any case gives the slowest growth in the magnitude of w_2 as $z \to \infty$. Condition (iii) needs to be satisfied by w_1 only for those values of k corresponding to lee waves which persist as $x \to \infty$. To put it in another way, for lee waves which do not vanish as x, (iii) must be satisfied because the ground is supposed to be level as $x \to \infty$. For these lee waves, then,

$$-\coth \mu_1 h = \frac{\mu_2}{\mu_1}. \tag{198}$$

If $l_2 > l_1$, there is no solution for (198), whatever the value (real) of k. Hence there are no discrete lee waves unless $l_1 > l_2$. If $l_1 > l_2$, (198) has no solution if $k > l_1$, for its left-hand side is negative and its right-hand side positive. For $l_1 > k > l_2$,

$$\mu_1 = \sqrt{k^2 - l_1^2} = -i\sqrt{l_1^2 - k^2},$$

and

$$-\coth \mu_1 h = \coth i\sqrt{l_1^2 - k^2}\, h = -i \cot \sqrt{l_1^2 - k^2}\, h.$$

Hence (198) becomes

$$\sqrt{l_1^2 - k^2} \cot \sqrt{l_1^2 - k^2}\, h = -\sqrt{k^2 - l_2^2}. \tag{199}$$

(This differs from Scorer's equation immediately preceding his equation (17) by the negative sign on the right-hand side.) Equation (199) can have a solution only if

$$(2n + 1 + \tfrac{1}{2})\pi \geq \sqrt{l_1^2 - k^2}\, h > (2n + \tfrac{1}{2})\pi,$$

or, since $k > l_2$, if

$$\sqrt{l_1^2 - l_2^2}\, h \geq (2n + \tfrac{1}{2})\pi. \tag{200}$$

(This agrees with Scorer's (17). There must therefore be a misprint in the equation preceding his (17).) If (200) is satisfied for $n = 0, 1, \ldots,$ and $N - 1$, there are N discrete lee waves. If $k < l_2$, again there is no solution for (198), for the left-hand side is imaginary and the right-hand side real. Thus k must be between l_1 and l_2.

For a barrier of the shape

$$\zeta_0(x) = \frac{b}{1 + (x/a)^2},$$

we recognize that

$$\zeta_0 = ab \int_0^\infty \cos kx e^{-ka}\, dk. \tag{201}$$

Realizing, as before, that $\zeta(x, z)$ satisfies (181), as well as w, and with (201) in mind, we can write down the solution as

$$\zeta = ab \int_0^\infty \frac{f(k, z)}{f(k, -h)} \cos kx e^{-ka}\, dk, \tag{202}$$

in which

$$f(k, z) = \begin{cases} e^{-\mu_2 z} & \text{for} \quad z \geq 0, \\ \cosh \mu_1 z - \dfrac{\mu_2}{\mu_1} \sinh \mu_1 z & \text{for} \quad 0 \geq z \geq -h. \end{cases}$$

At those values of k where (198) is satisfied, the path of integration for (202) is *below* the singularities, so that the terms corresponding to the residues of the singularities appear for $x > 0$ but not for $x < 0$, or that the corresponding discrete waves appear on the lee side only. The contour also goes below branch points, which may not be a zero of $f(k, -h)$. The main contribution in the paper of Scorer [1948] is the demonstration of discrete lee waves. The factor $\beta/2$ could have been included in λ_1 and λ_2, and the factor β^2 in μ. Whereas the neglect of β^2 is of minor importance, the neglect of $\beta/2$ in the λ's calls forth the difficulty at $z = \infty$, encountered by all authors dealing with an isothermal or polytropic atmosphere. Whatever function is chosen to be the solution of (189) there will always be values of k for which the function becomes infinite as z approaches infinity. At large values of z the linearization process ceases to be valid, however small a coefficient is assigned to the function. This difficulty can be overcome by assigning a finite height to the atmosphere and ignoring the effect of the fluid above that height, where a free-surface condition is imposed instead. But this is intellectually unsatisfying.

A treatment essentially identical with Scorer's was given by Zierep [1956]. Scorer's work has been further developed by Palm and Foldvik [1960], who treated the coefficients in the governing differential equation as variable. Other works on lee waves are listed in the bibliography.

Three-dimensional lee waves have been studied by Scorer and Wilkinson [1956], Crapper [1959 and 1962], and Drazin [1961]. The problems studied by them are intimately related to the problem of internal-wave resistance encountered by submerged bodies moving in a stratified fluid. Scorer and Wilkinson superposed ridges inclined at various angles to the wind to produce an isolated hill. Crapper used the method of singularities, which is akin to the method of solution by the Green's function. Drazin treated the case of an incompressible fluid for small values of the Froude number F (or small values of the parameter $F^2 = U^2/g\beta h^2$, h being a reference length) everywhere in the fluid. The first approximation is a horizontal flow with the vertical component of the vorticity equal to zero (see Chapter 1, Section 4). The second approximation is then found by adding a term of the order of F^2 to the first approximation, as determined by the governing equations. The approximation, however, is not uniformly valid. In one case treated by Drazin, for instance, it breaks down at the top of a hemisphere placed in the stratified stream.

The absence of internal waves upstream from an obstacle placed in a stratified stream was demonstrated by Crapper [1959], essentially with Rayleigh's approach of using a small friction, as has been mentioned. This absence can also be demonstrated by the flow establishment approach, i.e., by solving an initial-value problem and letting the time approach infinity, as was done for surface or internal waves by Stoker [1953b], Palm [1953], and Engevik [1971]. Engevik [1975] finally showed, by using the Laplace transform, the equivalence of these two approaches.

17. INTERNAL WAVES IN BASINS OR CHANNELS OF VARIABLE DEPTH

In basins or channels of variable depth, the wave motion of a stratified fluid is governed by a partial differential equation instead of an ordinary differential equation, since the vertical coordinate cannot be separated from the two horizontal ones for basins of general shape, or from one of these coordinates even for channels with constant cross sections. In this section we shall first present some comparison theorems concerning the frequency, and then give, by way of concrete examples, some solutions for internal-wave motion in a circular channel fully filled with a continuously stratified fluid or partially filled with two superposed fluid layers.

17.1. *Comparison Theorems for Internal Waves in Basins or Channels of Variable Depth*

Consider first the case of a finite three-dimensional domain V occupied by a stratified fluid. Since the depth is now variable the kinematic condition at the rigid boundary of V cannot be conveniently expressed in terms of the vertical velocity component w. It is better to derive the partial differential equation that governs the motion in terms of a new variable.

Let the velocity components in the directions of increasing x, y, and z be denoted by u, v, and w, and let p and ρ be the perturbations in pressure and in density, respectively. The direction of increasing z is, as before, opposite to that of the gravitational acceleration. The linearized equations of motion are

$$\bar{\rho} u_t = -p_x \tag{203}$$

$$\bar{\rho} v_t = -p_y, \tag{204}$$

$$\bar{\rho} w_t = -p_z - g\rho. \tag{205}$$

We shall consider an incompressible fluid. The linearized equation of incompressibility is

$$\rho_t + w\bar{\rho}' = 0, \tag{206}$$

where the prime indicates differentiation with respect to y. The equation of continuity (5) then has the form

$$u_x + v_y + w_z = 0. \tag{207}$$

Again, the time factor $\exp(-i\sigma t)$ will be assumed for all perturbation quantities. Then

$$i\sigma\bar{\rho}u = p_x, \qquad i\sigma\bar{\rho}v = p_y, \tag{208}$$

and combination of (205) and (206) gives

$$w = -\frac{i\sigma p_z}{\sigma^2\bar{\rho} + g\bar{\rho}'}. \tag{209}$$

Substituting (208) and (209) into (207), we have

$$p_{xx} + p_{yy} + \sigma^2\bar{\rho}\frac{\partial}{\partial z}\frac{p_z}{\sigma^2\bar{\rho} + g\bar{\rho}'} = 0. \tag{210}$$

If the direction numbers of the normal to a stationary rigid boundary are (l, m, n), the boundary condition there is

$$lu + mv + nw = 0. \tag{211}$$

At a free surface the pressure is constant. But the pressure variation has two parts: the perturbation pressure p and the variation of the mean pressure due to the free-surface displacement (denoted by ζ). Hence the condition on pressure at the free surface is

$$p - g\bar{\rho}\zeta = 0. \tag{212}$$

The kinematic condition at the free surface is

$$w = \zeta_t. \tag{213}$$

Combination of (209), (212), and (213) gives

$$p = \frac{g\bar{\rho}p_z}{\sigma^2\bar{\rho} + g\bar{\rho}'}. \tag{214}$$

It is well known, and it has been shown, that if $\bar{\rho}'$ is never positive σ^2 is positive (i.e., σ is real), and that, for truly internal waves, the inequality (162) holds. Indeed, it follows from (216) below when we let $p_1 = p_2$.

We shall now show [Yih, 1976a]

Theorem 1. *σ^2 increases if $\bar{\rho}$ is reduced by a constant or if both $\bar{\rho}$ and $\bar{\rho}'$ (henceforth assumed negative) are everywhere decreased, that is, if $\bar{\rho}$ is everywhere decreased in such a way that $-\bar{\rho}'$ is everywhere increased.*

To prove this theorem let us compare two wave motions, one with

$$\rho = \rho_1, \qquad p = p_1, \qquad \sigma = \sigma_1$$

and the other with

$$\bar{\rho} = \rho_2 \leq \rho_1, \qquad p = p_2, \qquad \sigma = \sigma_2, \qquad \rho_2' \leq \rho_1' < 0,$$

where ρ_1 and ρ_2 are functions of z only. The two motions are near each other in the sense that ρ_2 is only slightly less than ρ_1 for all values of z, so that p_1 is nearly the same as p_2, and σ_1 nearly the same as σ_2.

We shall first treat the case in which no free surface or other density discontinuities exist. Consider the integral (with V denoting the fluid volume, as before)

$$\int_V \left[\left(\frac{1}{\rho_1} p_2 p_{1x}\right)_x + \left(\frac{1}{\rho_1} p_2 p_{1y}\right)_y + \sigma_1^2\left(\frac{p_2 p_{1z}}{\sigma_1^2 \rho_1 + g\rho_1'}\right)_z\right] dV, \tag{215}$$

and another in which the subscripts 1 and 2 are exchanged. Since p_1 satisfies (210) with σ replaced by σ_1 and $\bar{\rho}$ by ρ_1, and p_2 satisfies (210) with σ replaced by σ_2 and $\bar{\rho}$ by ρ_2, (215) gives, upon use of the Gauss-Green theorem,

$$i\sigma_1 \int_S p_2(lu_1 + mv_1 + nw_1)\, dS$$
$$= \int_V \left[\frac{1}{\rho_1}(p_{1x}p_{2x} + p_{1y}p_{2y}) + \frac{\sigma_1^2}{\sigma_1^2\rho_1 + g\rho_1'} p_{1z}p_{2z}\right] dV, \tag{216}$$

where S is the surface of V. From the integral which is (215) with subscripts 1 and 2 exchanged, we obtain the result

$$i\sigma_2 \int_S p_1(lu_2 + mv_2 + nw_2)\, dS$$
$$= \int \left[\frac{1}{\rho_2}(p_{1x}p_{2x} + p_{1y}p_{2y}) + \frac{\sigma_2^2}{\sigma_2^2\rho_2 + g\rho_2'} p_{1z}p_{2z}\right] dV. \tag{217}$$

The left-hand sides of (216) and (217) are zero because of (211), the condition at the rigid boundary. The difference between (216) and (217) is

$$\int_V \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)(p_{1x}p_{2x} + p_{1y}p_{2y})\, dV$$
$$+ \int_V \frac{\sigma_1^2\sigma_2^2(\rho_1 - \rho_2) + g(\sigma_2^2\rho_1' - \sigma_1^2\rho_2')}{(\sigma_2^2\rho_2 + g\rho_2')(\sigma_1^2\rho_1 + g\rho_1')} p_{1z}p_{2z}\, dV = 0. \tag{218}$$

Upon making ρ_2 close to ρ_1, p_1 is close to p_2, σ_1 is close to σ_2, etc. Then quantities like $p_{1x}p_{2x}$ and so on are positive everywhere except possibly near the stagnation points of the smaller basin, where their absolute values are small. Hence they are positive on the average. Then, since

$$0 < \rho_2 \leq \rho_1 \qquad \text{and} \qquad \rho_2' \leq \rho_1' < 0,$$

(218) shows that σ_2^2 must be greater than σ_1^2.

If there is a free surface, the left-hand sides of (216) and (217) are, upon the use of the free-surface boundary condition (214), with $\bar{\rho}$ identified with ρ_1 or ρ_2, σ identified with σ_1 or σ_2,

$$\int_{S'} \frac{\sigma_1^2 p_1 p_2}{g\rho_1}\, dS' \qquad \text{and} \qquad \int_{S'} \frac{\sigma_2^2 p_1 p_2}{g\rho_2}\, dS', \tag{219}$$

respectively, and the term

$$\int_{S'} \frac{p_1 p_2}{g} \left(\frac{\sigma_2^2}{\rho_2} - \frac{\sigma_1^2}{\rho_1} \right) dS' \tag{220}$$

must be added to the right-hand side of (218), with S' denoting that part of S that is the free surface. The conclusion that

$$\sigma_2^2 > \sigma_1^2 \tag{221}$$

then cannot be reached if $\rho_2 < \rho_1$. However, if there are internal density jumps and $\Delta\rho_1 < \Delta\rho_2$ whenever $\Delta\rho_1$ exists, layer-by-layer integration and the interfacial conditions show that (221) stands, provided that the density inequalities on the preceding page are satisfied.

Note that since the conclusion (221) is reached by a comparison of neighboring states, it is automatically guaranteed that wave motions of the same mode are being compared. Furthermore, the restriction of neighboring states can be removed by continuation *through neighboring states*, so that the same mode is maintained while the two motions finally compared are no longer in neighboring states, that is, ρ_1 is no longer near ρ_2. (Mathematically speaking, ρ_1 being near ρ_2 means $(\rho_1 - \rho_2)\rho_1^{-1} \ll 1$.)

Since for truly internal waves (31) holds, for small $\bar{\rho}'$ the vertical velocity at the free surface is very small, as can be seen from (214) and (209). Therefore, when studying truly internal waves for small $\bar{\rho}'$ we can treat the free surface as rigid, and replace (214) by

$$p_z = 0. \tag{222}$$

Consider now the practically important case of a straight channel with depth varying with y but not with x, which is measured longitudinally along the channel. The variations of σ^2 and c^2 with k^2 for such a channel, with k denoting the wave number in the x-direction and with $\sigma = kc$, have been investigated by Yih (1972b) and are given by

Theorem 2. *For a straight channel with depth varying with y but not with x,*

$$\frac{d\sigma^2}{d(k^2)} > 0 \qquad \textit{and} \qquad \frac{dc^2}{d(k^2)} < 0,$$

if k denotes the wave number in the x-direction.

The proof of Theorem 2 is similar to that of Theorem 1. All we need to do is to replace the term p_{xx} in (210) by $-k^2p$ and to proceed as before, with $\bar{\rho}$ kept invariant and with k^2 taking on two neighboring values k_1^2 and k_2^2. Again, by continuation k_1^2 does not have to be near k_2^2 when we use Theorem 2 to compare σ_1^2 with σ_2^2, or c_1^2 with c_2^2.

A somewhat more general theorem than Theorem 2 can be established in an entirely similar fashion for two basins 1 and 2 which can be made congruent (i.e., identical) by the transformations

$$x_2 = \alpha x_1, \qquad y_2 = \beta y_1, \tag{223}$$

if both α and β are greater than unity or both less than unity. We shall, without loss of generality, consider the former case.

Congruence implies that when (223) is satisfied by (x_1, y_1) and (x_2, y_2) not only are the depths of the channels the same but, since $\bar{\rho}(z)$ is invariant, the elevations z_1 and z_2 in the two basins at which $\bar{\rho}$ is the same are equal.

The result is given by

Theorem 3. *For wave motion of the same mode in a fluid with the same stratification in two basins that can be made congruent by the transformation* (223),

$$\sigma_2^2 > \sigma_1^2$$

if

$$\alpha > 1 \qquad \textit{and} \qquad \beta > 1.$$

We shall now assume $\bar{\rho}$ to be continuous, and attempt to see what effect a change of shape or size of the fluid-occupied domain of a basin has on the frequency of internal waves. For this purpose we need to use the Boussinesq approximation, which amounts to ignoring the variation of ρ except for the term $g\bar{\rho}'$ in (209) and (210). Whether we consider an exponentially stratified fluid with

$$\bar{\rho} = C\exp(-\beta z) \tag{224}$$

or a linearly stratified fluid with

$$\bar{\rho} = C(1 - \beta z), \tag{225}$$

after the Boussinesq approximation is made we can write (210) as

$$p_{xx} + p_{yy} - \lambda^2 p_{zz} = 0, \tag{226}$$

where

$$\lambda^2 = \frac{\sigma^2}{g\beta - \sigma^2}. \tag{227}$$

Let the boundary of the basin be given by

$$F(x, y, z) = 0. \tag{228}$$

For an open basin of the simplest shapes, z is a single-valued function of x and y; it can, however, be multivalued. For a closed basin flowing full, it is at least double-valued. The condition at the boundary, (211), can now be written as

$$F_x p_x + F_y p_y - \lambda^2 F_z p_z = 0. \tag{229}$$

Since for truly internal waves we can treat a free surface as a rigid surface and therefore a part of (228), we can absorb (222) into (229).

Inspection of (226) and (229) gives [Yih, 1976a].

Theorem 4. *With the actual* $\bar{\rho}$ *given by* (224) *or* (225) *maintained unchanged (i.e., with* βz *equal to* $\beta' z'$ *after a scale magnification or reduction), if any horizontal dimension of the basin is multiplied by* a *and the vertical dimension by* b, *i.e., if* x, y, *and* z *in* $F(x, y, z)$ *are replaced by*

$$\frac{x}{a}, \quad \frac{y}{a}, \quad \frac{z}{b},$$

then λ^2 *is multiplied by* b^2/a^2.

Hence if the widening exceeds the deepening, λ^2 (and so σ^2) is reduced; if the deepening exceeds the widening, σ^2 is increased; and if the magnification (or reduction) of scale is uniform in all directions, $a = b$, and σ^2 remains unchanged.

17.2. *Internal Waves in a Circular Channel*

Long lakes can often be replaced by a straight channel when one studies waves in them. If the depth of a lake is small in comparison with its width, its bottom can often be approximated by a circular arc.

Limnologists generally use a model of two superposed water layers, each homogeneous in itself, to approximate distribution in temperature and in density. In order to provide some information to limnologists I shall present here the results of calculations carried out by Yang and Yih [1976] for the frequencies (Fig. 7.1) of the first four sloshing modes of internal waves in two superposed fluid layers in a circular channel for different ratios of the depths of the layers and for two positions of the free surface. (See Fig. 7 for definitions of a, b, b', d_1, and d_2.) The flow patterns (Fig. 7.2) of the first four sloshing modes are also given, when the fluids occupy a semicircular space and the depth of the upper layer is one quarter of the radius of the channel.

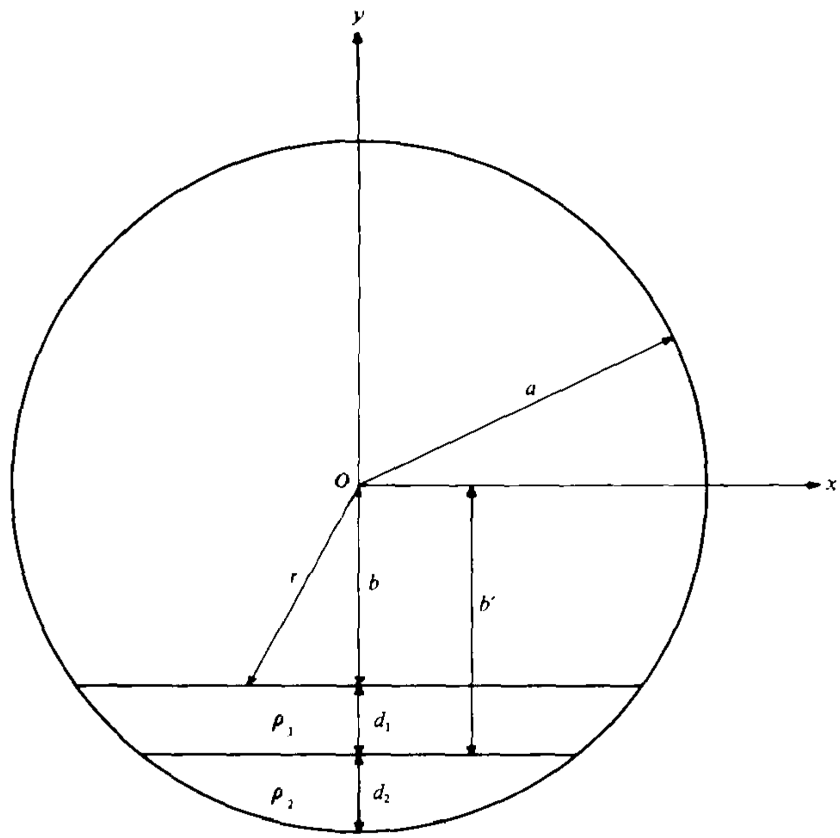


FIGURE 7. Definition sketch. (*J. Fluid Mech.*, *74*, *courtesy of the Cambridge Univ. Press.*)

We shall denote the density of the upper fluid by ρ_1 and that of the lower fluid by ρ_2. Since each layer is homogeneous within itself, we shall assume the flow in it to be irrotational, and denote the velocity potential for the upper layer by ϕ_1 and that of the lower layer by ϕ_2.

Then

$$\begin{aligned} \nabla^2\phi_1 &= 0, \\ \nabla^2\phi_2 &= 0, \end{aligned} \tag{230}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2}$$

x and y being Cartesian coordinates, with y measured in the direction of the vertical. The condition at the solid boundary of the channel is, for either layer,

$$\frac{\partial\phi}{\partial n} = 0, \tag{231}$$

where n is measured in a direction normal to the boundary.

At the free surface the pressure is constant. By combining the kinematic

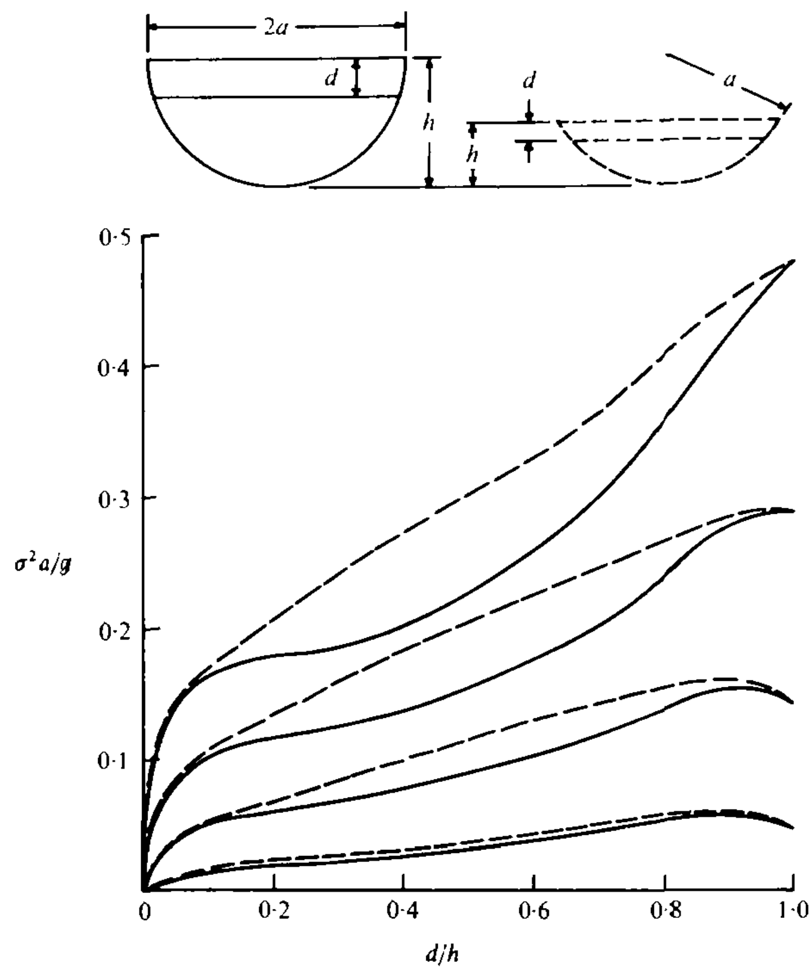


FIGURE 7.1. Results for $\sigma^2 a/g$ as a function of d/h. Solid line: $h = a$. Dotted line: $2h = a$. (*J. Fluid Mech.*, *74*, *courtesy the Cambridge Univ. Press.*)

condition with the Bernoulli equation we have the free-surface condition

$$\frac{\partial^2 \phi_1}{\partial t^2} + g \frac{\partial \phi_1}{\partial y} = 0.$$

If we assume the exponential factor exp $(-i\sigma t)$ for all perturbation quantities, this condition becomes

$$\sigma^2 \phi_1 = g(\phi_1)_y, \tag{232}$$

in which the subscripts t and y indicate partial differentiation.

On the interface the kinematic condition is

$$(\phi_1)_y = (\phi_2)_y \tag{233}$$

and the dynamic condition, obtained in a similar way as (232), is

$$\sigma^2(\rho_2 \phi_2 - \rho_1 \phi_1) = g(\rho_2 - \rho_1)(\phi_1)_y. \tag{234}$$

The differential system to be solved consists of (230) to (234), with σ^2 as the eigenvalue.

In our numerical calculations it is often convenient to use the stream

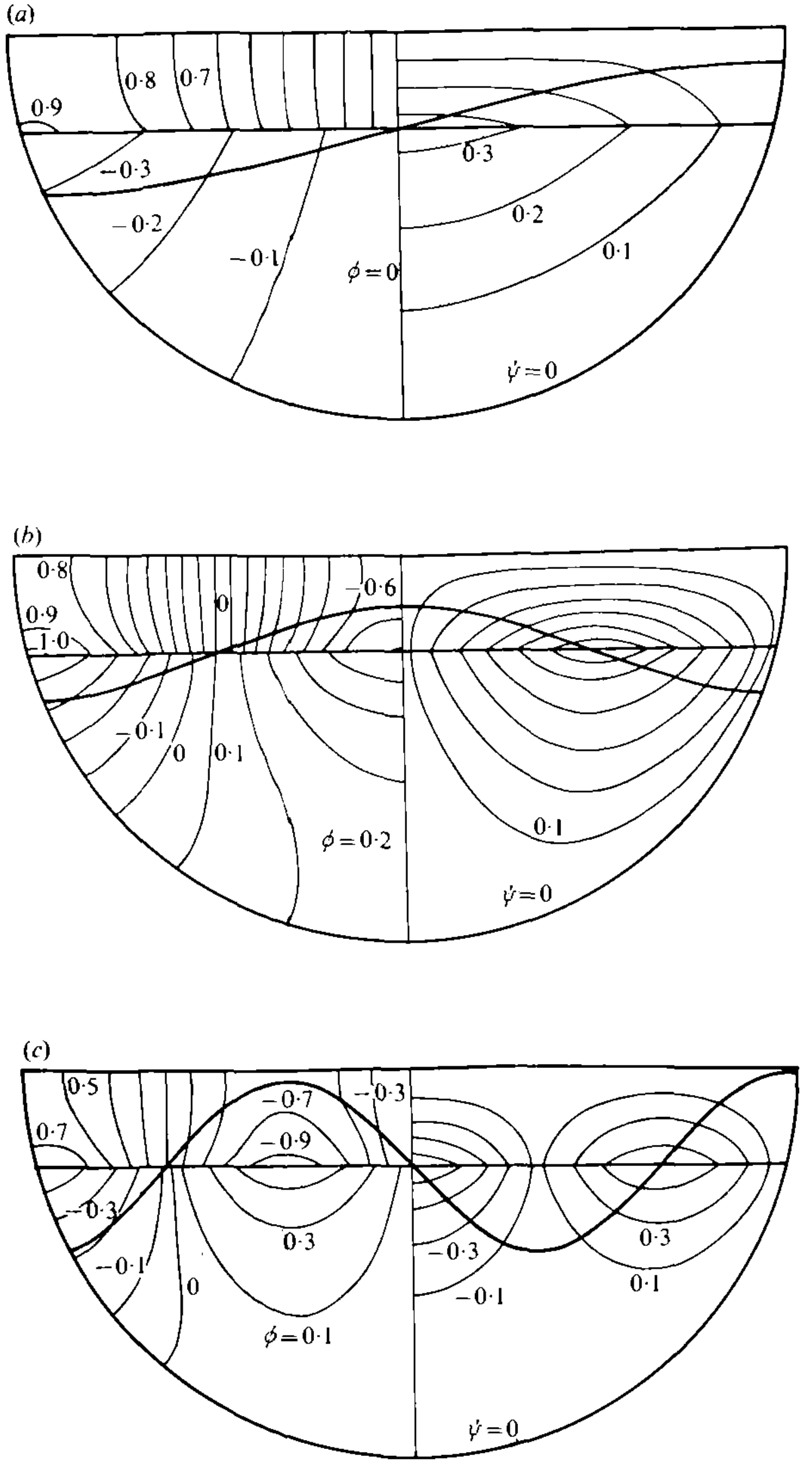


FIGURE 7.2. (a) Flow pattern for the first sloshing mode of internal waves. The intervals for the stream function ψ and the velocity potential ϕ are 0.1 throughout. The free surface is not assumed flat. The height of interface is exaggerated. $\sigma^2 a/g = 0.0152$. (b) Flow pattern for the second sloshing mode of internal waves. The intervals for the stream function ψ and the velocity potential ϕ are 0.1 throughout. The free surface is not assumed flat. The height of interface is exaggerated. $\sigma^2 a/g = 0.0465$. (c) Flow pattern for the third sloshing mode of internal waves. The intervals for the stream function ψ and the velocity potential ϕ are 0.2 throughout. The free surface is not assumed flat. The height of interface is exaggerated. $\sigma^2 a/g = 0.0794$.

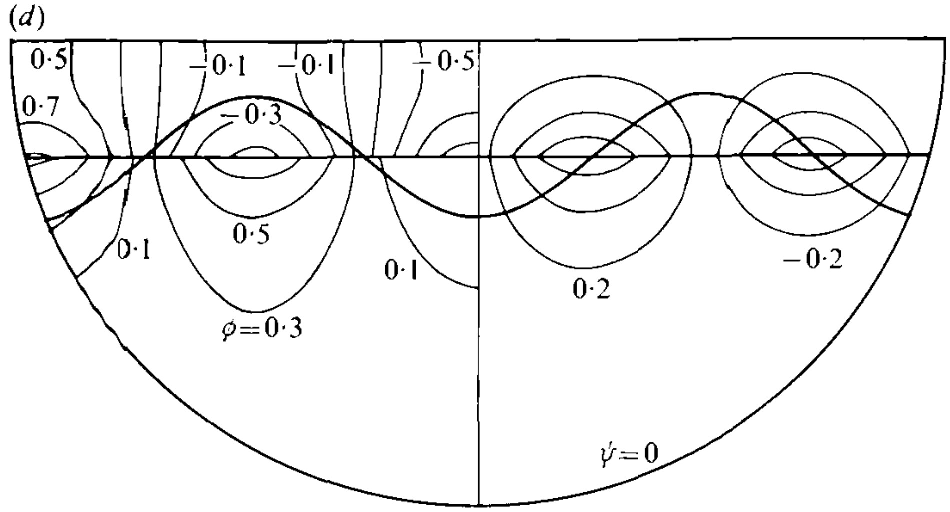


FIGURE 7.2. (d) Flow pattern for the fourth sloshing mode of internal waves. The intervals for the stream function ψ and the velocity potential ϕ are 0.2 throughout. The free surface is not assumed flat. The height of interface is exaggerated. $\sigma^2 a/g = 0.10843$. (*J. Fluid Mech., 74, courtesy of the Cambridge Univ. Press.*)

functions. In terms of the stream functions ψ_1 and ψ_2, with the subscript 1 indicating the upper layer and the subscript 2 indicating the lower layer, the differential system consists of the following equations:

$$\nabla^2\psi_1 = 0, \qquad \nabla^2\psi_2 = 0, \tag{235}$$

$$\psi_1 = 0 = \psi_2 \qquad \text{on the solid boundary,} \tag{236}$$

$$\sigma^2(\psi_1)_y = g(\psi_1)_{xx}, \qquad \text{on the free surface,} \tag{237}$$

$$\psi_1 = \psi_2 \qquad \text{on the interface,} \tag{238}$$

$$\sigma^2(\rho_2\psi_2 - \rho_1\psi_1)_y = g(\rho_2 - \rho_1)(\psi_1)_{xx} \qquad \text{on the interface.} \tag{239}$$

Conditions (237) and (239) are obtained by differentiating (232) and (234) with respect to x and using the Cauchy Riemann equations, and Eq. (238) follows directly from (233) by using the Cauchy Riemann equations.

We shall first give some exact results for the extreme cases in which either the upper layer or the lower layer is extremely thin, which cannot be conveniently obtained by numerical computations. These results will provide some check for the trend of the numerical results and some guidance to the curves representing them. For internal waves the free surface can be considered fixed.

For the case of an extremely thin upper layer, when the total maximum depth is fixed, the result is simply that for *internal* waves,

$$\sigma = 0.$$

The reasoning is briefly as follows. Compare the case with that of internal waves in a rectangular channel with the total depth equal to the maximum total depth of the circular channel, the upper-layer depth exactly equal to that for the case under consideration, and the width equal to the width of the interface. The σ for the circular channel can be shown to be less [Yih 1976a] than that for the rectangular channel, for the interfaces are of the same width and the excess area occupied by the thin upper fluid in the circular channel over that in the rectangular channel is negligibly small, whereas the domain of the lower fluid in the rectangular channel contains that in the circular channel. For *internal* waves the upper surface can be taken as flat, and the frequency for the rectangular channel is, from a simple calculation with (230), (231), (233), and (234),

$$\sigma^2 = g_1' \tanh kd_1,$$

where d_1 is the depth of the upper fluid,

$$g_1' = \frac{\Delta\rho}{\rho_1} g, \qquad \Delta\rho = \rho_2 - \rho_1,$$

and

$$k = \frac{2n+1}{b}\pi \qquad \text{or} \qquad \frac{2n\pi}{b},$$

b being with width of the rectangular channel and n a positive integer. It is clear then that the σ for the rectangular channel approaches zero as d_1 approaches zero, and *a fortiori* the σ for the circular channel must approach zero. This calculation also illustrates the well known fact that when either layer is thin we can replace g by $g\Delta\rho/(\rho$ of thin layer), and ignore the existence of the deep layer. This fact will be recalled when we calculate σ for the case of a thin lower layer in the following paragraphs.

For the case of extremely thin lower layer, we note first that, as has been noted, when the lower layer is very thin the frequency for any mode of internal waves is the same as that for the same mode of free-surface waves for one single layer (the thin lower layer), as if the upper fluid did not exist, if g is changed to

$$g' = \frac{\rho_2 - \rho_1}{\rho_2} g. \tag{240}$$

Thus to find the σ for the case of the very thin lower layer, we need only to find the σ for free-surface waves on that layer, but with g replaced by g'.

For the first sloshing mode, we take

$$\phi_2 = \theta = -\arctan \frac{x}{y},$$

which satisfies (230) and (231). On the fictitious free surface, which is really the interface, where $y = -b'$, we have, after some simple calculations and using (232), with ϕ_2 replacing ϕ_1,

$$\sigma^2 = \frac{g'}{b'},$$

if terms of $O[(x/b')^3]$ are neglected. Note that x'/b is small if the lower layer is thin, or $a - b' \ll a$, radius of circular channel. In the limit,

$$\sigma^2 = \frac{g'}{a}. \tag{241}$$

Solutions for higher modes can be similarly obtained. But solutions for all modes can be obtained once for all by adapting Budiansky's solutions [1960] for a single thin layer, by merely replacing his g by our g'. The long-wave equation used by Budiansky is, after replacement of ϕ_2 by ϕ and g by g',

$$\frac{\partial}{\partial x}(h_2 \phi_x) + \frac{\sigma^2}{g'} \phi = 0, \tag{242}$$

where

$$h_2 = (a^2 - x^2)^{1/2} - b' \doteq (a - b') - \frac{x^2}{2a}.$$

With

$$\xi^2 = 2a(a - b')x^2,$$

(242) can be written as

$$\frac{\partial}{\partial \xi}[(1 - \xi^2)\phi_{,1}] + \frac{2a\sigma^2}{g'} \phi = 0, \tag{243}$$

which is Legendre's equation. In order not to have any singularities at $\xi^2 = 1$, it is necessary that

$$\frac{2\sigma^2 a}{g'} = n(n + 1), \tag{244}$$

n being an integer representing a mode. For $n = 1$ (244) gives (241), as expected.

The analytical results obtained above for a very thin upper or lower layer provide the end points (at $d/h = 0$ and $d/h = 1$) in Fig. 7.1. The intermediate values for $\sigma^2 a/g$ were obtained by numerical computation, as were the flow patterns given in Fig. 7.2.

17.3. *Internal Waves in a Circular or Elliptic Pipe*

We shall now present solutions for internal waves in a stratified incompressible fluid contained in a circular or elliptic pipe [Yih, 1975a]. Let the pipe axis be horizontal and the distance measured along it be denoted by x. The vertical distance measured from the axis of the pipe will be denoted by z, and the y-axis is perpendicular to the x- and z-axes.

Let the density distribution be given by (224) or (225). If the Boussinesq approximation is made, the governing equation is (226), with λ given by (227).

If b and c are the semiaxes of the elliptic cross section of the pipe, the F in (228) is

$$F = \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1. \tag{245}$$

Setting

$$p = f(y, z) \exp(ikx), \tag{246}$$

we can rewrite (226) as

$$f_{yy} - \lambda^2 f_{zz} - k^2 f = 0 \tag{247}$$

and the boundary condition (229) as

$$\frac{yf_y}{b^2} - \lambda^2 \frac{zf_z}{c^2} = 0 \qquad \text{on} \quad F = 0. \tag{248}$$

The task is to solve (247) and (248), which constitute an eigenvalue problem. The solution provides the eigenfunction f and the eigenvalue λ^2. Knowing λ^2, we can use (227) to obtain σ. Then, with p given by (246), we can use (208) and (209) to obtain the velocity components.

We shall first show that the problem of the elliptic cross section can be reduced to a problem of a circular cross section. Let

$$z' = \frac{bz}{c} \qquad \text{and} \qquad \lambda' = \frac{b\lambda}{c}, \tag{249}$$

which is essentially the same transformation as that made by Høiland [1962] for rotating fluids in elliptic channels. Then (245) becomes

$$F = \frac{y^2}{b^2} + \frac{z'^2}{b^2} - 1, \tag{250}$$

and (247) and (248) become

$$f_{yy} - \lambda'^2 f_{z'z'} - k^2 f = 0 \tag{251}$$

and

$$yf_y - \lambda'^2 z' f_{z'} = 0 \qquad \text{on} \quad F = 0, \tag{252}$$

respectively. Dropping the primes on z' and λ', we have

$$f_{yy} - \lambda^2 f_{zz} - k^2 f = 0, \tag{253}$$

$$y f_y - \lambda^2 z f_z = 0 \qquad \text{on the pipe boundary,} \tag{254}$$

the pipe boundary being $y^2 + z^2 = b^2$. Equations (253) and (254) constitute the differential system for internal waves in a circular pipe under the stated assumptions on the mean density $\bar{\rho}$. After solving the problem for a circular cross section, we can write λ' and z' for the λ and z in (253) and (254), and use (249) to compute the true λ for the case of the elliptic cross section, as well as the eigenfunction in terms of y and the true z. Thus we need only to study the case of the circular cross section, as one can indeed expect, knowing Theorem **4** in Section 17.1.

We shall, then, deal with (253) and (254) only. We can simplify things further by measuring all lengths, including the wavelength, in units of b. Then (253) and (254) retain their form, but the pipe boundary is now

$$y^2 + z^2 = 1. \tag{255}$$

The geometry (now circular) of the problem requires a transformation of coordinates so as to allow a separation of variables one or other of which is constant on the circle given by (255). The required transformation is

$$y = \frac{\cos\mu\cos\nu}{\cos\alpha}, \qquad z = \frac{\sin\mu\sin\nu}{\sin\alpha}, \tag{256}$$

where α is defined by

$$\frac{\sigma}{\sqrt{g\beta}} = \cos\alpha, \quad \text{or } \lambda^{-1} = \tan\alpha, \qquad \text{with } \alpha < \frac{\pi}{2}. \tag{257}$$

The transformation (256) is the same as the one employed by Barcilon [1968] to study *two-dimensional* waves in rotating fluids, although the definition of α is necessarily different.

Although Barcilon's transformation was used by him to study transverse oscillations only, it is equally useful for studying three-dimensional waves. The area inside the circle (255) is mapped many times over into rectangles in the μ-ν plane. We shall choose the rectangle

$$-\alpha \leq \mu \leq \alpha, \qquad \alpha \leq \nu \leq \pi - \alpha \tag{258}$$

as the domain for our analysis. By virtue of (257), (253) can be written as

$$f_{yy} - (\cot^2\alpha) f_{zz} - k^2 f = 0, \tag{259}$$

which, upon the use of (256), becomes

$$(\cos^2\alpha)(f_{\mu\mu} - f_{\nu\nu}) - \frac{k^2}{2}(\cos 2\nu - \cos 2\mu) f = 0. \tag{260}$$

The circle (255) is, in the μ-ν domain, the boundary of the rectangle (258). That is, it is transformed onto four line segments:

$$\mu = \alpha, \qquad \alpha \le \nu \le \pi - \alpha,$$
$$\mu = -\alpha, \qquad \alpha \le \nu \le \pi - \alpha,$$
$$\nu = \alpha, \qquad -\alpha \le \mu \le \alpha,$$
$$\nu = \pi - \alpha, \qquad -\alpha \le \mu \le \alpha.$$

The boundary condition (254) is

$$\frac{\partial f}{\partial \mu} = 0 \qquad \text{at} \quad \mu = \pm\alpha \tag{261}$$

on the first two segments and

$$\frac{\partial f}{\partial \nu} = 0 \qquad \text{at} \quad \nu = \alpha \quad \text{and at} \quad \nu = \pi - \alpha \tag{262}$$

on the second two segments.

Substituting

$$f = F(\mu)G(\nu) \tag{263}$$

into (260) and separating variables, we have

$$F'' + (a - 2q\cos 2\mu)F = 0, \tag{264}$$

$$G'' + (a - 2q\cos 2\nu)G = 0, \tag{265}$$

with

$$q = -\frac{k^2}{2\cos^2\alpha}. \tag{266}$$

Equations (264) and (265) have identical forms. The quantity a is a constant arising from the separation of variables. These are in the canonical form of the Mathieu equation. We now make the important observation that y and z given by (256) have the period 2π in μ and ν, and any solutions of (264), and (256), to have a unique value at any point (y, z), must also have the same period* 2π in μ and ν. This condition determines a as a function of q. There are infinitely many characteristic values for a for each value of q [McLachlan, 1947]:

$$a_0 = -\tfrac{1}{2}q^2 + \tfrac{7}{128}q^4 + O(q^6),$$
$$b_1 = 1 - q - \tfrac{1}{8}q^2 + \tfrac{1}{64}q^3 + O(q^4), \qquad a_1 = b_1(-q),$$
$$b_2 = 4 - \tfrac{1}{12}q^2 + O(q^4), \qquad a_2 = 4 - \tfrac{5}{12}q^2 + O(q^4),$$
$$b_3 = 9 + \tfrac{1}{16}q^2 - \tfrac{1}{64}q^3 + O(q^4), \qquad a_3 = b_3(-q),$$

* Or the period $2\pi N$, N being an integer.

and so on. It is important to note that the first term of a_n or b_n is n^2, the eigenfunction associated with a_n is denoted by ce_n, and that associated with b_n is denoted by se_n. The function $ce_{2p}(\mu, q)$ (p = integer) contains the functions $\cos 2m\mu$, m being an integer, and the function $ce_{2p+1}(\mu, q)$ contains the functions $\cos (2m + 1)\mu$. Similarly, $se_{2p}(\mu, q)$ contains $\sin 2m\mu$ and se_{2p+1} contains $\sin (2m + 1)\mu$. It is well known, and it can be easily demonstrated by starting with $q = 0$, that $ce_n(\mu, q)$ has $2n$ internal zeros in $(0, 2\pi)$, and $se_n(\mu, q)$ has $2n - 1$ internal zeros in $(0, 2\pi)$, but vanishes at $\mu = 0$ and $\mu = 2\pi$. What has been said of the functions of μ is of course also true for the functions of v.

The solution of (260), (261), and (262) is of the following two classes:

$$f = ce_n(\mu, q)ce_n(v, q), \qquad n = 0, 1, 2, \ldots, \tag{267}$$

$$f = se_n(\mu, q)se_n(v, q), \qquad n = 1, 2, 3, \ldots. \tag{268}$$

The eigenvalues α are determined by (261):

$$ce_n'(\mu, q) \equiv \frac{d}{d\mu} ce_n(\mu, q) = 0 \qquad \text{for} \quad \mu = \pm\alpha \tag{269a}$$

or

$$se_n'(\mu, q) \equiv \frac{d}{d\mu} se_n(\mu, q) = 0 \qquad \text{for} \quad \mu = \pm\alpha. \tag{269b}$$

Because

$$ce_n'(\alpha, q) = -ce_n'(-\alpha, q), \; se_n'(\alpha, q) = se_n'(-\alpha, q),$$

we can impose (269a) and (269b) for $\mu = \alpha$ only. Also, in regard to v,

$$ce_n'(\alpha, q) = (-1)^{n+1} ce_n'(\pi - \alpha, q),$$
$$se_n'(\alpha, q) = (-1)^n se_n'(\pi - \alpha, q),$$

so that (262) is automatically satisfied if (261) is for $\mu = \alpha$. We must, however, obviously rule out

$$\alpha = 0 \qquad \text{and} \qquad \alpha = \pi/2,$$

which are roots of (269a). The actual determination of α depends on n and q and is in general rather complicated. In the following paragraphs we shall determine α for long waves.

We shall solve the differential system consisting of (261) to (265) by expansion in power series in k^2. Since we have seen that it suffices to solve (264) together with

$$F'(\alpha) = 0, \tag{270}$$

and that F and G are identical functions of the variables μ and ν, we shall assume

$$F = F_0 + k^2F_1 + k^4F_2 + \cdots,$$
$$\alpha = \alpha_0 + k^2\alpha_1 + k^4\alpha_2 + \cdots.$$

Substituting these into (264) and collecting terms of the zeroth order in k, we have [see (264) and (265) for $q = 0$]

$$a = n^2 \qquad (n = \text{integer}),$$

and the solutions correspond exactly to those found by Barcilon (the only difference being in the relation of σ with α), and are of two classes [see again (264) and (265) for $q = 0$]:

$$\text{(i)} \qquad f_0 = \cos n\mu \cos n\nu, \qquad \alpha_0 = \frac{m\pi}{n},$$
$$n = 3, 4, \ldots, \qquad m = 1, 2, \ldots, N^{(1)}, \qquad \text{with} \quad N^{(1)} = \begin{cases} \frac{n}{2} - 1 & \text{for } n \text{ even}, \\ \frac{n-1}{2} & \text{for } n \text{ odd}. \end{cases} \tag{271}$$

$$\text{(ii)} \qquad f_0 = \sin n\mu \sin n\nu, \qquad \alpha_0 = \frac{2m+1}{2n}\lambda\pi;$$
$$n = 2, 3, \ldots, \qquad m = 1, 2, \ldots, N^{(2)}, \qquad \text{with} \quad N^{(2)} = \begin{cases} \frac{n}{2} - 1 & \text{for } n \text{ even}, \\ \frac{n-1}{2} - 1 & \text{for } n \text{ odd}. \end{cases} \tag{272}$$

The values of n are so determined as to rule out 0 and $\pi/2$ as eigenvalues for α_0, which would correspond to λ being zero or infinity, respectively.

Collecting terms of order k^2 in (264), and recalling that, for class (i), $F_0 = \cos n\mu$, we have

$$f_1'' + n^2 f_1 = -\frac{g\beta}{\sigma_0^2}\cos 2\mu \cos n\mu,$$

the solution of which is

$$f_1 = -\frac{g\beta}{8\sigma_0^2}\left[\frac{\cos(n-2)\mu}{n-1} - \frac{\cos(n+2)\mu}{n+1}\right].$$

Application of (270) to the order k^2 then produces

$$\alpha_1 = -\frac{n^2-2}{2n^2(n^2-1)}\tan\alpha_0.$$

A similar calculation for class (ii) gives

$$f_1 = \frac{g\beta}{8\sigma_0^2}\left[\frac{\sin(n+2)\mu}{n+1} - \frac{\sin(n-2)\mu}{n-1}\right],$$

$$\alpha_1 = -\frac{n^2-2}{2n^2(n^2-1)}\tan\alpha_0.$$

Recall that α_0 for class (i) is different from α_0 for class (ii).

Further calculation for higher orders in k^2 can be performed without any difficulty. We note that $\sigma/\sqrt{g\beta} = \cos\alpha$ increases with k, since α decreases with k.

To visualize the flow pattern, we first consider the two-dimensional flow (without x-variation) represented by (271) or (272). For this two-dimensional flow let

$$(v, w) = (V, W). \tag{273}$$

Then

$$\bar{\rho}V = f_y, \qquad \bar{\rho}W = -\lambda^2 f_z, \tag{274}$$

and (207), (208), and (209), with $k = 0$, and (253) lead to

$$\frac{\partial(\bar{\rho}V)}{\partial y} + \frac{\partial(\bar{\rho}W)}{\partial z} = 0, \tag{275}$$

which is not exactly right but is consistent with the Boussinesq approximation. Then we obtain (ψ being the stream function)

$$\bar{\rho}V = \psi_z = f_y, \qquad -\bar{\rho}W = \psi_y = \lambda^2 f_z, \tag{276}$$

which allows us to compute ψ once f is known. For instance, if $n = 2$ and $m = 0$ in (272), we have (since $f = f_0$ and $\alpha = \alpha_0$ for two-dimensional flow), from (256),

$$f = 2yz.$$

From (276) it follows then

$$\psi = y^2 + z^2 - 1, \tag{277}$$

where the constant of integration -1 is to make $\psi = 0$ on the circle. Similarly, if $n = 3$ and $m = 0$ in (272),

$$\psi = y(y^2 + z^2 - 1); \tag{278}$$

if $n = 3$ and $m = 1$ in (271),

$$\psi = z(y^2 + z^2 - 1); \tag{279}$$

and if $n = 4$ and $m = 1$ in (271),

$$\psi = yz(y^2 + z^2 - 1). \tag{280}$$

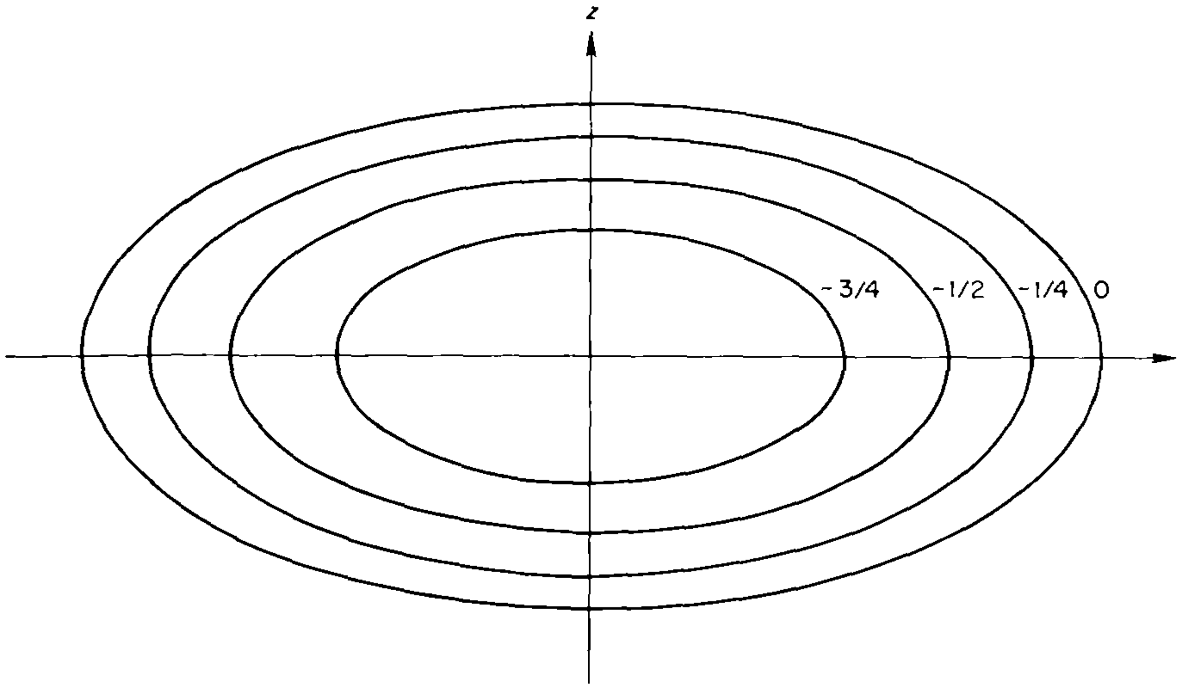
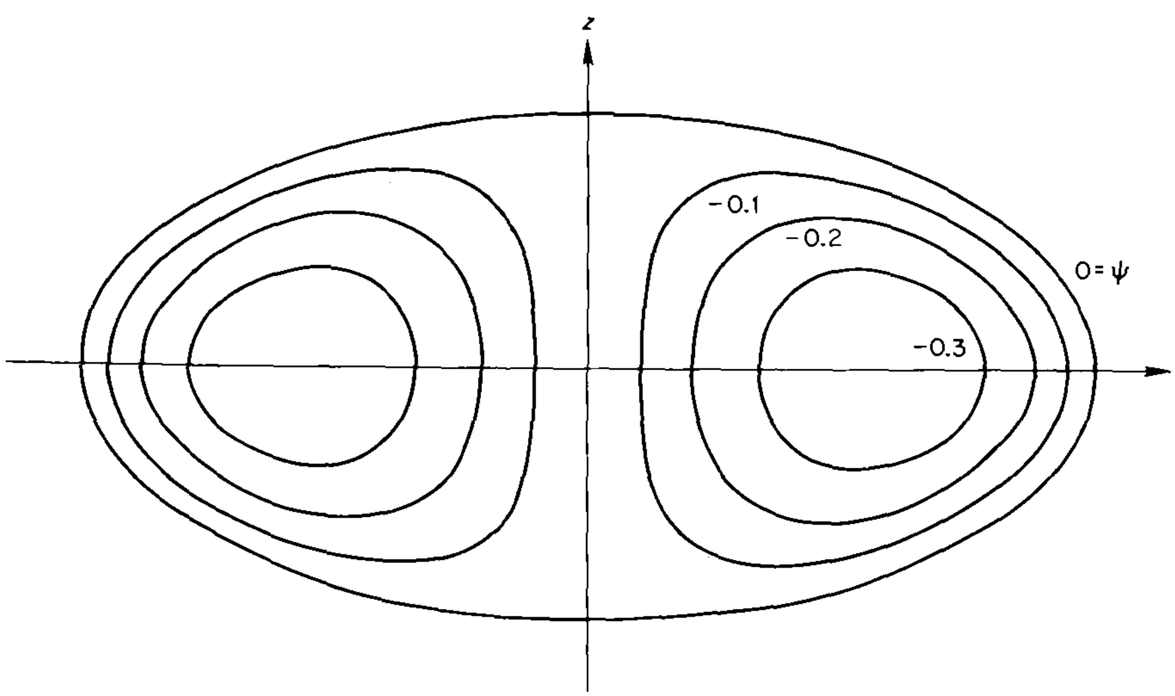
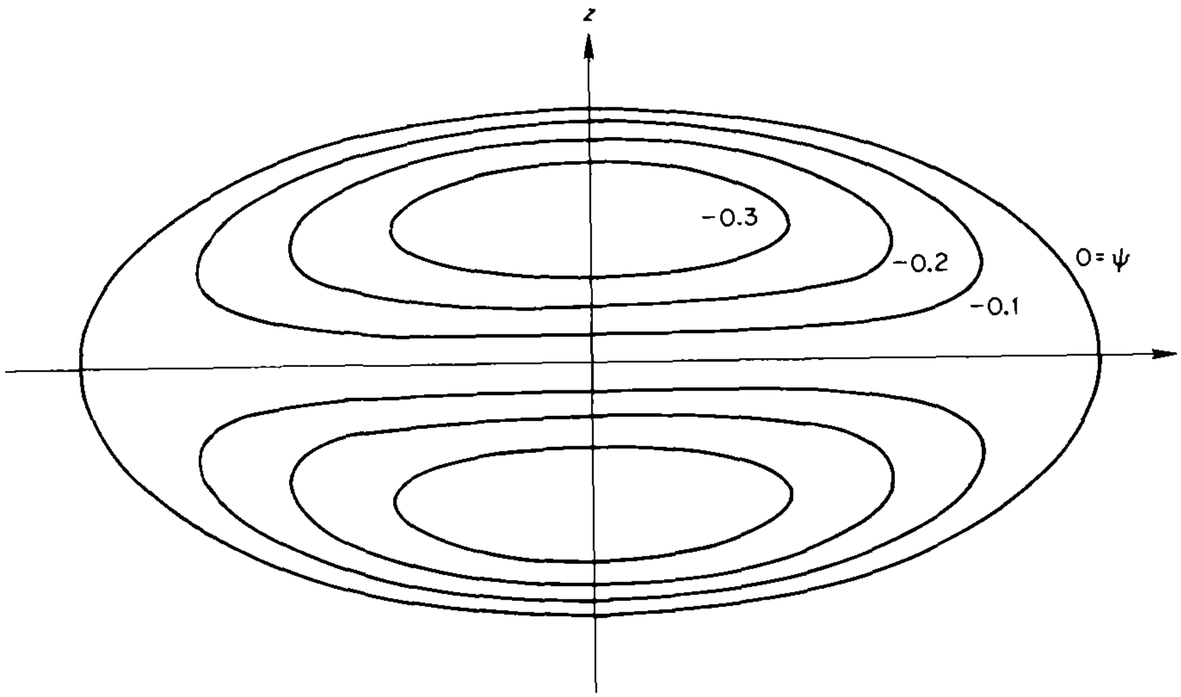


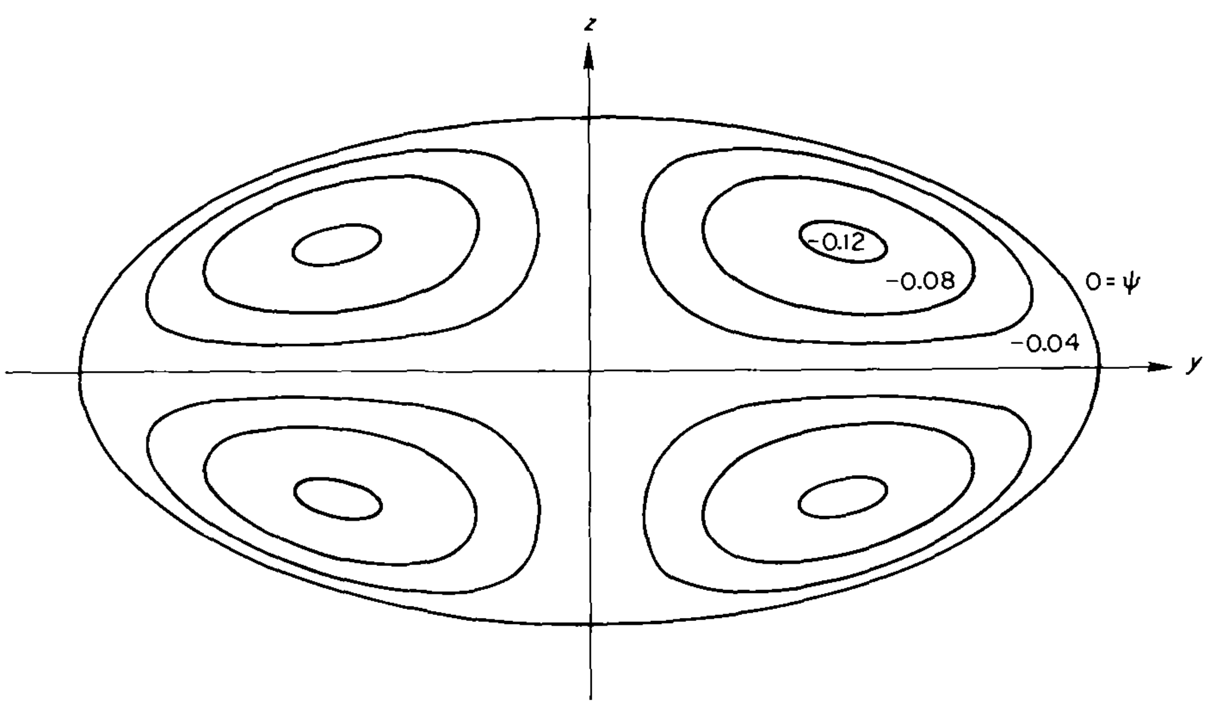
FIGURE 7.3. Streamline pattern for the two-dimensional sloshing mode of internal-wave motion in an elliptic channel, with the ratio of semiaxes equal to $\frac{1}{2}$. (a) The stream function is given by Eq. (277a), and the interval for ψ is -0.25.



(b) The stream function is given by Eq. (278a), and the interval for ψ is -0.1.



(c) The stream function is given by Eq. (279a), and the interval for ψ is -0.1.



(d) The stream function is givën by Eq. (280a), and the interval for ψ is -0.04.

For an ellipse, we can use (249) to obtain the stream function corresponding to (277) through (280). These are

$$\psi = y^2 + \frac{z^2}{c'^2} - 1, \tag{277a}$$

$$\psi = y\left(y^2 + \frac{z^2}{c'^2} - 1\right), \tag{278a}$$

$$\psi = \frac{z}{c'}\left(y^2 + \frac{z^2}{c'^2} - 1\right), \tag{279a}$$

$$\psi = \frac{yz}{c'}\left(y^2 + \frac{z^2}{c'^2} - 1\right), \tag{280a}$$

in which again y and z are measured in units of b, and c' is the semiaxis in the z-direction divided by the semiaxis b in the y-direction. The flow patterns represented by (277a) to (280a) are shown in Figs. 7.3a through d. It should be kept in mind that the motion is not steady but oscillatory. When waves propagate in the x-direction, one can visualize the motion by superposing the flow patterns in Figs. 7.3a through d on a three-dimensional flow sinusoidally varying with x.

NOTES

Section 16

1. Rayleigh's device of using a small frictional force to eliminate upstream surface waves produced by a barrier placed in a subcritical stream, extended to stratified fluids by Crapper [1959], has been shown by Engevik [1975], by the use of Laplace transforms, to be equivalent to the approach of establishment of flow used by Stoker [1953b] and others.

2. Using the Boussinesq approximation and adopting Oseen's approach, though allowing the wind (horizontal) velocity to vary as a function of the vertical coordinate, Booker and Bretherton [1967] showed that when the Richardson number R is greater than $\frac{1}{4}$ internal waves are attenuated by a factor $\exp[-2\pi(R - \frac{1}{4})^{1/2}]$ as they pass through a critical layer, at which the wind velocity is equal to the phase velocity of the waves or to zero if the flow passes over a stationary wavy boundary. This attenuation still exists, though the factor thereof is modified, if viscous and diffusive effects are taken into account, as shown by Hazel [1967].

Margolis and Su [1975] considered stratified flows over a fixed barrier. These flows are supposed to have a critical layer, at which the velocity is zero, and nonlinearity must be taken into account in order to describe the flows adequately.

3. Under the assumption that the horizontal scale of a mountain profile is much larger than its vertical scale and the vertical wavelengths of steady internal gravity waves, Drazin and Su [1975] showed that $\zeta(x, z)$, the displacement of the streamline at (x, z) above its undisturbed level, can be expressed as

$$\zeta(x, z) = \zeta_0(x) \operatorname{Re}\left[\frac{f(z)}{f(0)}\right] + \operatorname{Im}\left[\frac{f(z)}{\pi f(0)}\right] P \int_{-\infty}^{\infty} \frac{\zeta_0(t)}{t - x}\, dt.$$

In this formula x is the horizontal coordinate in the direction of the wind, with mean velocity $\bar{u}(z)$ at $x = -\infty$, z is the vertical coordinate, $z = \zeta_0(x)$ describes the mountain profile, P indicates the principal value, and $f(z)$ satisfies the linearized long-wave equation

$$\frac{d}{dz}\left(\bar{\rho}\bar{u}^2 \frac{df}{dz}\right) - g \frac{d\bar{\rho}}{dz} f = 0$$

and the condition of upward energy propagation as $z \to +\infty$, i.e., the Sommerfeld radiation condition at $z = +\infty$. The undisturbed density distribution is given by $\bar{\rho}(z)$, and g denotes the gravitational acceleration.

In the formula of Drazin and Su, the effects of the atmospheric structure embodied in $f(z)$, and of the mountain profile, embodied in $\zeta_0(x)$, are explicitly and separately calculated (though the results are multiplied).

4. Dissipative effects on stratified shear flows over an obstacle was studied by Su [1975]. He concludes that in the neighborhood of the obstacle viscous and diffusive effects can be ignored, whereas higher up in the flow they are not negligible. This conclusion supports the usual lee-wave theories for inviscid and nondiffusive fluids in that near the obstacles these theories are not much in error. However, separation has not been taken into account.

Readers of this paper by Su may find puzzling his statement that his (6) reduces to his (11), which is a differential equation of a lower order. His equation (16) seems to have the right order and to represent (6). I think there must be a misprint in (11).

5. Meyer [1973] studied unidirectional waves propagating in a horizontal direction in a linearly stratified fluid, using the Boussinesq approximation but taking viscosity and diffusivity fully into account. The conclusions reached are reassuring: Waves decaying in the propagation direction, presumably identifiable with windward waves generated by an obstacle in a stratified wind, decay very fast (and infinitely fast in the inviscid limit), whereas waves decaying in the direction opposite to the propagation direction, presumably identifiable with lee waves behind an obstacle placed in the wind, decay slowly, and in the inviscid limit do not decay at all. Hence the classical lee-wave theories are reaffirmed.

Readers of Meyer's paper should be aware that there are misprints in his dispersion relation (5). However, the subsequent two equations, containing the final results, are correct if the absolute-value signs are replaced by simple parentheses.

General

1. A long paper by Longuet-Higgins [1965] deals with the longer-period response of an unbounded, stratified ocean of uniform depth to initial distributions of wind stress applied at the surface. The β-plane approach is used. That is, the variation of the horizontal components of the Coriolis force is assumed to be that in a plane tangent to the earth at a point (say at the bottom of the ocean) around which the phenomenon of fluid flow is under study.

Chapter 3

STEADY FLOWS OF FINITE AMPLITUDE

1. INTRODUCTION

Certain large-amplitude flows of an incompressible fluid with variable density or of a compressible fluid with variable entropy can be treated exactly. These flows will be discussed in detail in this chapter, with special emphasis on the role played by gravity.

Potential flows with a free surface or an interface separating one fluid from the other are flows with an extreme form of stratification. These are also presented in detail, as well as internal hydraulic jumps and gravity currents. Finally, unidirectional flows of stratified viscous fluids are discussed. All the flows presented in this chapter are profoundly affected by gravity.

2. GOVERNING EQUATION FOR TWO-DIMENSIONAL FLOWS OF AN INCOMPRESSIBLE AND INVISCID FLUID

For two-dimensional steady flows of an incompressible fluid, the equation of continuity is

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0,$$

in which, in accordance with (1.18),

$$(u', w') = \sqrt{\frac{\rho}{\rho_0}}\,(u, w), \tag{1}$$

and ρ_0 is a reference density. Thus a stream function ψ' for the associated flow exists, and

$$u' = \frac{\partial \psi'}{\partial z}, \qquad w' = -\frac{\partial \psi'}{\partial x}. \tag{2}$$

Although the velocity components u_i were introduced in (1.18) to show the correspondence between actual and associated flows if gravity effects are neglected, they are still useful when gravity effects are taken into account, because they embody the inertial effects of density variation once and for all. The equations of motion are, in terms of u' and w',

$$\rho_0\left(u'\frac{\partial u'}{\partial x} + w'\frac{\partial u'}{\partial z}\right) = -\frac{\partial p}{\partial x}, \tag{3}$$

$$\rho_0\left(u'\frac{\partial w'}{\partial x} + w'\frac{\partial w'}{\partial z}\right) = -\frac{\partial p}{\partial z} - g\rho. \tag{4}$$

If the y-component of the vorticity of the associated flow is denoted by η',

$$\eta' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} = \nabla^2\psi', \tag{5}$$

and (3) and (4) become

$$\rho_0\eta'\frac{\partial\psi'}{\partial x} = \frac{\partial}{\partial x}\left(p + \frac{\rho_0(u'^2 + w'^2)}{2}\right), \tag{6}$$

$$\rho_0\eta'\frac{\partial\psi'}{\partial z} = \frac{\partial}{\partial z}\left(p + \frac{\rho_0(u'^2 + w'^2)}{2}\right) + g\rho. \tag{7}$$

Multiplying (6) by dx and (7) by dz and adding the results, we have

$$\rho_0\eta'\,d\psi' = d\left(p + \frac{\rho(u^2 + w^2)}{2}\right) + g\rho\,dz = dH - gz\,d\rho, \tag{8}$$

in which

$$H = p + \frac{\rho(u^2 + w^2)}{2} + g\rho z \tag{9}$$

is the Bernoulli quantity, which is constant along a streamline, but may vary from streamline to streamline. Thus H, like ρ, is a function of ψ' alone, and (8) can be written as [Yih 1958]

$$\nabla^2\psi' + \frac{gz}{\rho_0}\frac{d\rho}{d\psi'} = \frac{1}{\rho_0}\frac{dH}{d\psi'} = h(\psi'), \tag{10}$$

which is the governing equation sought. Equation (10) is a modified (and improved) form of an equation due to Dubreil Jacotin [1935] and Long [1953b]:

$$\nabla^2\psi + \frac{1}{\rho}\frac{d\rho}{d\psi}\left(\frac{\psi_x^2 + \psi_z^2}{2} + gz\right) = f(\psi), \tag{11}$$

in which subscripts indicate partial differentiation.

The functions $d\rho/d\psi'$ and $dH/d\psi'$ are to be determined from upstream conditions, and represent the effects of the variation of specific weight and the variation of specific energy on fluid motion. The inertia effect of density variation is already embodied in ψ'.

Although (10) provides a powerful springboard for further studies in the mechanics of a nonhomogeneous incompressible fluid, it unfortunately has been derived under the restriction of steady flow. Since a theorem corresponding to the wonderful theorem of persistence of irrotationality (for steady or unsteady flows alike) does not exist at present, we do not know what a flow will eventually be like if it has been started from rest. In particular, this means that for steady flows we do not know what condition is the correct one to be assumed at infinity, at least for two-dimensional flows, for which gravity may have, under the assumption of inviscidness, the striking effect of rendering the velocity nonuniform at infinity, at very low speeds of a moving obstacle. This question has very direct bearings on the problem of fluid separation, as will be seen in Sections 6 and 8.

Thus we have good reason to regret that Nature has not been particularly generous here, and that although (10) fully determines the flow once the upstream conditions are known, these cannot be found without a study of the establishment of motion. Deprived of a convenient tool to study unsteady flows, we are no more able to determine readily the upstream condition than a historian is able to answer why a king is a king (other than that he was born one), if he has been deprived of the opportunity to study significant historical events.

An equation, like (10) but for unsteady flow, is thus much to be desired. But since this does not exist at the moment, in an analytical study at least we have to do what we can with (10). Just as in a study of rotational flows of a homogeneous inviscid fluid, we shall not ask how the upstream condition has come to be, but shall assume convenient and realistic ones and study the consequences. It is comforting that much can be achieved in this manner.

3. TYPES OF SOLUTIONS FOR STEADY FLOWS OF AN INCOMPRESSIBLE FLUID OF VARIABLE DENSITY

If the upstream conditions are arbitrarily specified, the functions $d\rho/d\psi'$ and $dH/d\psi'$ are usually not linear functions of ψ', and the solution of (10) is very difficult. To find exact solutions of (10), one may either try different upstream conditions and see whether these will make (10) linear, or else assume the equation to be linear forthwith and find the corresponding upstream conditions afterwards. The second approach is at once exhaustive and more economical, and will be adopted [Yih, 1960b].

If (10) is linear, it must have the general form

$$\nabla^2\psi' + z(C_0 + C_1\psi') = C_2 + C_3\psi', \tag{12}$$

in which C_0, C_1, C_2, and C_3 are constants. There are three classes.

Class I: $C_1 = C_3 = 0$. For this class, (12) assume the form

$$\nabla^2\psi' + C_0 z = C_2. \tag{13}$$

Class II: $C_1 = 0$, $C_2 = 0$. For this class, after ψ' has been changed by a constant, (12) becomes

$$\nabla^2\psi' + C_0 z = C_3\psi'. \tag{14}$$

Class III: $C_1 \neq 0$, $C_3 = 0$. For this class, (12) becomes

$$\nabla^2\psi' + C_1 z\psi' = C_2, \tag{15}$$

after ψ' has been changed by a constant. The case $C_1 \neq 0$ and $C_3 \neq 0$ is not new, since both z and ψ' can be changed by a constant to make (12) assume the form (15).

4. CLASS I: PSEUDOPOTENTIAL FLOWS

For $C_1 = C_3 = 0$, the solution of (13) is of the form

$$\psi' = \psi_a' - \frac{C_0 z}{2}\left[\frac{\alpha z^2}{3} + (1-\alpha)x^2\right] + \frac{C_2}{2}[\beta z^2 + (1-\beta)x^2], \tag{16}$$

in which ψ_a' satisfies the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\psi_a' = 0. \tag{17}$$

Since ψ' is the sum of a polynomial in z and a potential function, the flows represented by it will be called pseudopotential flows. Since C_0 is $(g/\rho_0)/(d\rho/d\psi')$, the density ρ is simply $C_0\rho_0\psi'/g$ plus a constant.

As a first example, consider the flow of a stratified fluid between two horizontal boundaries at $z = 0$ and $z = d$ into a two-dimensional sink at the origin. Since the flow is symmetric with respect to the z-axis, which is a streamline, one can consider the flow in the domain

$$-\infty < x \leq 0, \qquad 0 \leq z \leq d.$$

With U_m' denoting the weighted mean (weighted by $\sqrt{\rho/\rho_0}$) velocity at infinity, and with

$$\xi = \frac{x}{d}, \qquad \eta = \frac{z}{d} \tag{18}$$

as the new spatial variables, a distribution for u' at infinity which will allow (10) to reduce to (13) is

$$U' = 6U_m'\eta(1-\eta). \tag{19}$$

Actually, if U' takes the form of any polynomial of the second degree in η, (10) is reduced to (13). Equation (16) then has the form

$$\psi' = \psi_a' - U_m'd(2\eta^3 - 3\eta^2). \tag{20}$$

If

$$\Psi = \frac{\psi'}{U_m'd}, \qquad \Psi_a = \frac{\psi_a'}{U_m'd},$$

the boundary conditions are

$$\Psi = 0 \quad \text{at} \quad \eta = 0;$$
$$\Psi = 1 \quad \text{at} \quad \eta = 1 \quad \text{and at} \quad \xi = 0, \qquad 0 < \eta \leq 1;$$
$$\Psi = -2\eta^3 + 3\eta^2 \quad \text{at} \quad \xi = -\infty.$$

In terms of Ψ_a, the boundary conditions are

$$\Psi_a = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \eta = 1;$$
$$\Psi_a = 1 + 2\eta^3 - 3\eta^2 \quad \text{at} \quad \xi = 0, \qquad 0 < \eta \leq 1;$$
$$\Psi_a = 0 \quad \text{at} \quad \xi = -\infty.$$

Solution of (17) — with ψ_a' changed to Ψ_a — by the method of separation of variable yields

$$\Psi_a = \sum_{n=1}^{\infty} A_n e^{n\pi\xi} \sin n\pi\eta,$$

with

$$A_1 = \frac{2}{\pi}, \qquad A_2 = \frac{1}{\pi}\left(1 + \frac{3}{\pi^2}\right), \qquad A_3 = \frac{2}{3\pi}, \qquad A_4 = \frac{1}{\pi}\left(\frac{1}{2} + \frac{3}{8\pi^2}\right),$$

$$A_5 = \frac{2}{5\pi}, \qquad \text{etc.}$$

The final solution is

$$\psi' = U_m'd(\Psi_a + 3\eta^2 - 2\eta^3). \tag{21}$$

The density at infinity is given by

$$\rho = \rho_0 + (\rho_1 - \rho_0)(3\eta^2 - 2\eta^3), \tag{22}$$

in which ρ_0 and ρ_1 are the density at $\eta = 0$ and that at $\eta = 1$, respectively. The flow pattern has an eddy at the upper right corner. In this eddy the flow does not originate at infinity, and therefore (13) and (21) do not apply. However, the flow outside of the eddy should be approximately given by (21), if (22) and (19) are realized.

Another example is furnished by a two-dimensional flow of a stratified fluid over a lower boundary constituted by (in polar coordinates)

$$r = r_0, \quad 0 \leq \theta \leq \pi; \qquad r \geq r_0, \quad \theta = 0 \quad \text{and} \quad \theta = \pi.$$

No upper boundary is specified. After the solution is obtained, any streamline above the lower boundary can be taken to be the upper boundary. Here again ψ' is of the type

$$\psi' = \psi'_a - \frac{C_0 z^3}{6} + \frac{C_2}{2} z^2 \qquad \left(\text{with} \quad C_0 = \frac{g}{\rho_0} \frac{d\rho}{d\psi'}\right). \tag{23}$$

For the purpose of illustration, consider the case

$$\psi' = \psi'_a - \frac{C_0}{6} z^3 .$$

The solution satisfying

$$\nabla^2 \psi' + C_0 z = 0 \tag{24}$$

and

$$\psi' = 0 \qquad \text{on the lower boundary}$$

is

$$\psi' = \frac{C_0}{24} r_0^3 \left[3\left(\frac{1}{r_1} - r_1^3\right) \sin\theta - \left(\frac{1}{r_1^3} - r_1^3\right) \sin 3\theta \right], \tag{25}$$

with $r_1 = r/r_0$. The flow pattern is shown in Fig. 8. The stagnation points are

$$(r, \theta) = (\infty, 0), \quad (\infty, \pi), \quad (r_0, 0), \quad (r_0, \pi), \quad (r_0, \pi/6), \quad (r_0, 5\pi/6).$$

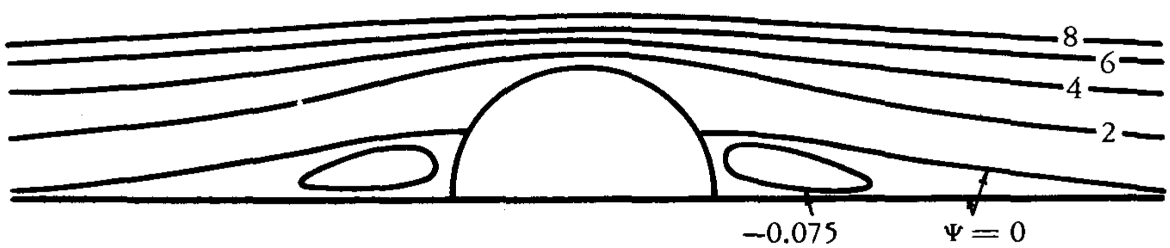


FIGURE 8. A two-dimensional pseudopotential flow. (*J. Fluid Mech., 9, part 2. Courtesy of the Cambridge Univ. Press.*)

Another solution of (24), corresponding to

$$\psi' = \psi'_a - \frac{C_0}{6} z^3 + C_4 z, \tag{26}$$

is

$$\psi' = \psi'_1 + C_4 r_0 \left(r_1 - \frac{1}{r_1} \right) \sin\theta,$$

in which ψ'_1 is the ψ' in (25).

Since ψ'_a is harmonic, one can use (16) to construct as many solutions as one pleases. Many of the exact and approximate methods used for the solution of potential-flow problems can be applied here, especially the method of singularities and the (inverse) method of analytic functions.

5. CLASS II: CHANNEL FLOWS AND LARGE-AMPLITUDE LEE WAVES

Perhaps the most interesting and important of the three classes of flows governed exactly by a linear equation is Class II. If a flow between two plane boundaries at a distance d apart is again considered, and if again ξ and η are defined by (18), (14) becomes

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\psi' + C_0\, d^3\eta = C_3\, d^2\psi', \tag{27}$$

in which

$$C_0 = \frac{g}{\rho_0}\frac{d\rho}{d\psi'} \tag{28}$$

is negative if the density stratification is stable and if the flow is from left to right. If U' is a reference velocity, division of (27) by $U'd$ yields

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\Psi + A\eta = B\Psi, \tag{29}$$

in which

$$\Psi = \frac{\psi'}{U'd}, \qquad A = \frac{C_0 d^2}{U'}, \qquad B = C_3 d^2.$$

Now if C_0 is negative, A is also negative, and from (28) it can be seen that $-(1/A)$ is really the square of a modified Froude number, because

$$-A = \frac{g'd}{U'^2}, \qquad g' = -\frac{g}{\rho_0}\frac{d\rho}{d\Psi}. \tag{30}$$

Thus A will be denoted by $-F^{-2}$, and (29) becomes

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\Psi - F^{-2}\eta = B\Psi, \tag{31}$$

in which F is the Froude number $U'/\sqrt{g'd}$. If the density stratification is stable but the flow is from right to left, C_0 and A are positive. But Ψ will be everywhere negative except possibly in the eddies. In any case the sign of Ψ can be changed, and (31) is again obtained. Naturally, whether the flow is from the right or the left should make no difference to the flow. We shall assume that the flow is always from left to right. As to the number B, it can be either positive or negative.

The solution of (31) is of the form

$$\Psi = \Psi_1 + \Psi_2, \tag{32}$$

with

$$\Psi_1 = -(BF^2)^{-1}\eta + K\cosh\sqrt{B}\eta + K'\sinh\sqrt{B}\eta, \tag{33}$$

$$\Psi_2 = \sum_{n=1}^{\infty} A_n \exp\left[\pm(B + n^2\pi^2)^{1/2}\xi\right]\sin n\pi\eta. \tag{34}$$

The solution Ψ_1 gives the upstream flow at infinity, where the flow is parallel. Whether B is positive or negative, the constants K and K' can always be adjusted to make Ψ_1 positive throughout the range $0 \leq \eta \leq 1$. Equation (33) possesses enough flexibility to fit any smooth upstream velocity distribution with uniformly small errors in the same range. [See Yih 1960b* and Long 1958.] However, since we have assumed, for the sake of linearity, that

$$\frac{g}{\rho_0} \frac{d\rho}{d\psi'} = C_0 + C_1 \psi', \tag{35}$$

the variation of ρ far upstream is dependent on that of ψ', hence on that of Ψ_1'. Thus the upstream distribution of ρ has to satisfy (35) before the governing equation can be linear, even if the upstream flow can be described by (33). The same remark applied to all three classes of flow under consideration. In the present class $C_1 = 0$, hence (35) has the form

$$\frac{d\rho}{d\psi'} = \text{constant far upstream.}$$

The next section will be devoted to channel flows of a stratified fluid, which have some engineering applications. The two sections following the next will deal with large-amplitude flows with lee waves, or waves in the lee of barriers, which are similar to atmospheric waves in the lee of mountain ranges, and with the phenomenon of blocking. These three sections should logically be numbered 5.1, 5.2, and 5.3, but because of their importance will be numbered 6, 7, and 8. They all pertain to flows of Class II.

6. CLASS II (*continued*): TWO-DIMENSIONAL STRATIFIED FLOW INTO A SINK

The two-dimensional flow of a stratified fluid [Yih, 1958] between two horizontal planes into a line sink serves as a simple example to show the striking effect of stratification. The reference length is the distance d between the plane boundaries. The dimensionless variables ξ and η defined by (18) will again be used, and the sink is located at $\xi = 0$ and $\eta = 0$.

It is convenient to halve the strength of the sink and consider the flow region

$$-\infty < \xi \leq 0, \qquad 0 \leq \eta \leq 1.$$

If far upstream

$$\Psi = \Psi_1 = \eta \tag{36}$$

and

$$\rho = \rho_0 - (\rho_0 - \rho_1)\eta, \tag{37}$$

in which ρ_0 is the density at $\eta = 0$ and ρ_1 the density at $\eta = 1$, the quantity $d\rho/d\psi'$ is constant, and the left-hand side of (10) is linear and in dimensionless

* Equation (10), from which all linear cases follow immediately, already appeared in a paper [Yih, 1958] presented to the Third U.S. National Congress of Applied Mechanics in June, 1958, before they were explicitly discussed in either of these papers.

terms assumes the form of the left-hand side of (31). Furthermore, from (36) (for $\xi = -\infty$) it follows that the B in (31) is exactly $-F^{-2}$. Thus the governing equation is

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\Psi - F^{-2}\eta = -F^{-2}\Psi. \tag{38}$$

In fact, from (33) and (36) it can be seen that $K = K' = 0$, and $B = -F^{-2}$.

Before the boundary conditions and the solution are presented, equations (36) and (37) should be interpreted. Equation (37) states simply that the density variation with respect to z (or η) is linear. The linear variation is the simplest, and can be realized in experiments. Furthermore, any stable stratification occurring in nature can be approximated by a linear variation in ρ in a first approximation. Equation (36) states that the pseudovelocity $U' = \partial\psi'/\partial z$ is constant at $\xi = -\infty$, or that

$$\sqrt{\frac{\rho}{\rho_0}}\, U = \text{constant}. \tag{39}$$

For small variation in ρ, U is practically constant. This represents again a simple situation. If in an actual flow U is not inversely proportional to $\sqrt{\rho}$, the solution based on a constant $\sqrt{\rho}\, U$ can be used as a first approximation. Yih [1958] has shown that if the flow issues horizontally from a large reservoir, $\sqrt{\rho}\, U$ is indeed a constant.

The boundary conditions are

$$\begin{aligned}
&\Psi = 0 \quad &&\text{at} \quad \eta = 0,\\
&\Psi = 1 \quad &&\text{at} \quad \eta = 1 \quad \text{and at} \quad \xi = 0, \quad \eta \neq 0,\\
&\Psi = \eta \quad &&\text{at} \quad \xi = -\infty.
\end{aligned}$$

In terms of Ψ_2 these become

$$\begin{aligned}
&\Psi_2 = 0 \quad &&\text{at} \quad \eta = 0 \quad \text{and} \quad \eta = 1,\\
&\Psi_2 = 1 - \eta \quad &&\text{at} \quad \xi = 0, \quad \eta \neq 0,\\
&\Psi_2 = 0 \quad &&\text{at} \quad \xi = -\infty.
\end{aligned}$$

The solution for $F > 1/\pi$ is

$$\Psi = \eta + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left[(n^2\pi^2 - F^{-2})^{1/2}\xi\right]\sin n\pi\eta, \tag{40}$$

in which the coefficients $2/\pi n$ are the coefficients A_n in (34), and have been determined from

$$1 - \eta = \sum_{n=1}^{\infty} A_n \sin n\pi\eta.$$

The solution is valid so long as $F > 1/\pi$. The solution for a different location of the sink differs slightly from (40), but still has the property that

$\Psi \to \eta$ as $\xi \to -\infty$, provided $F > 1/\pi$. Thus, any attempt at separating one part of the fluid from the rest is futile at Froude numbers much* greater than $1/\pi$. The flow patterns for $F = 0.32$, 0.35, 0.5 and ∞ are shown in Figs. 9. The flow pattern for $F = \infty$ is identical with that for a homogeneous fluid.

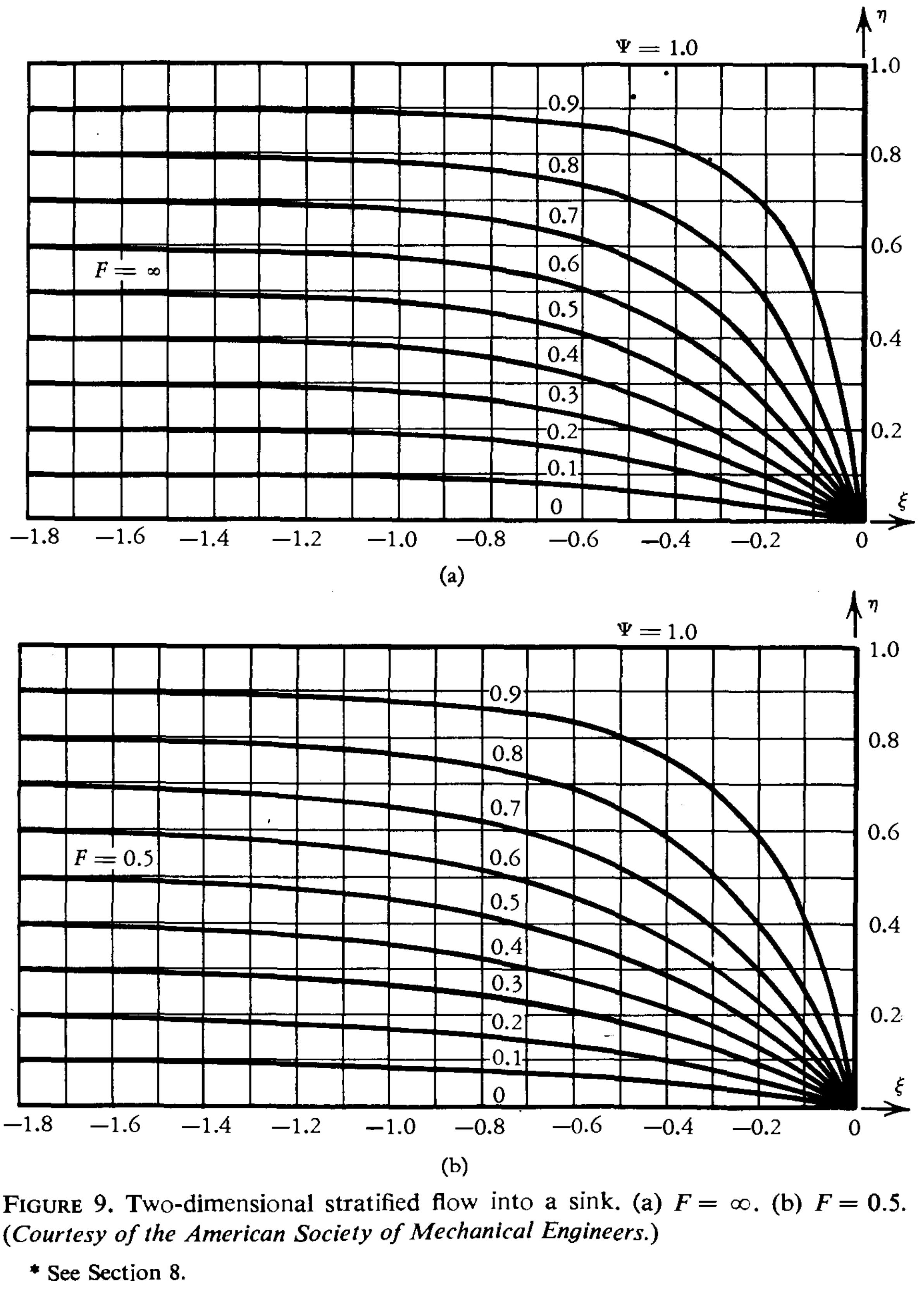


FIGURE 9. Two-dimensional stratified flow into a sink. (a) $F = \infty$. (b) $F = 0.5$. (*Courtesy of the American Society of Mechanical Engineers.*)

* See Section 8.

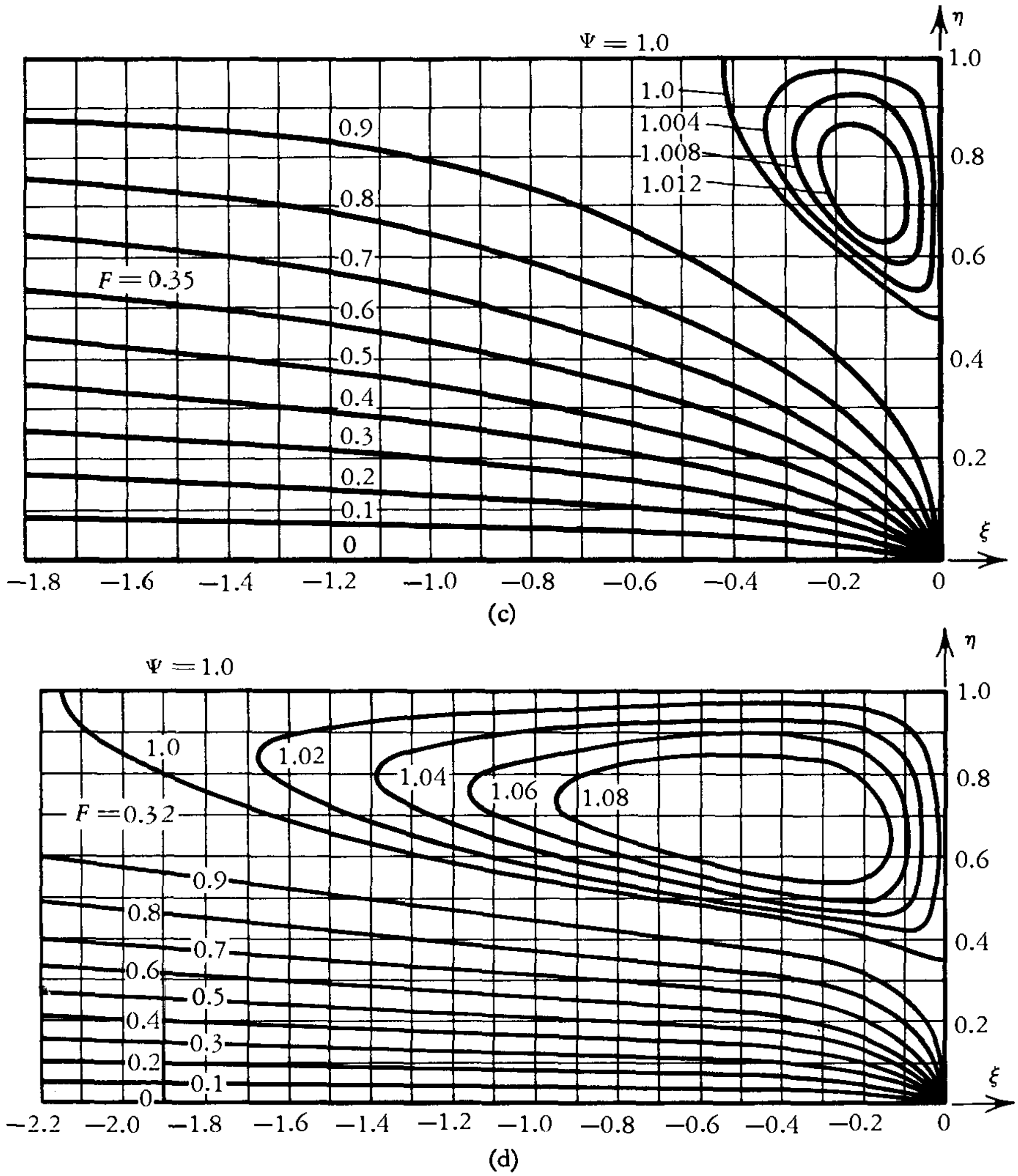
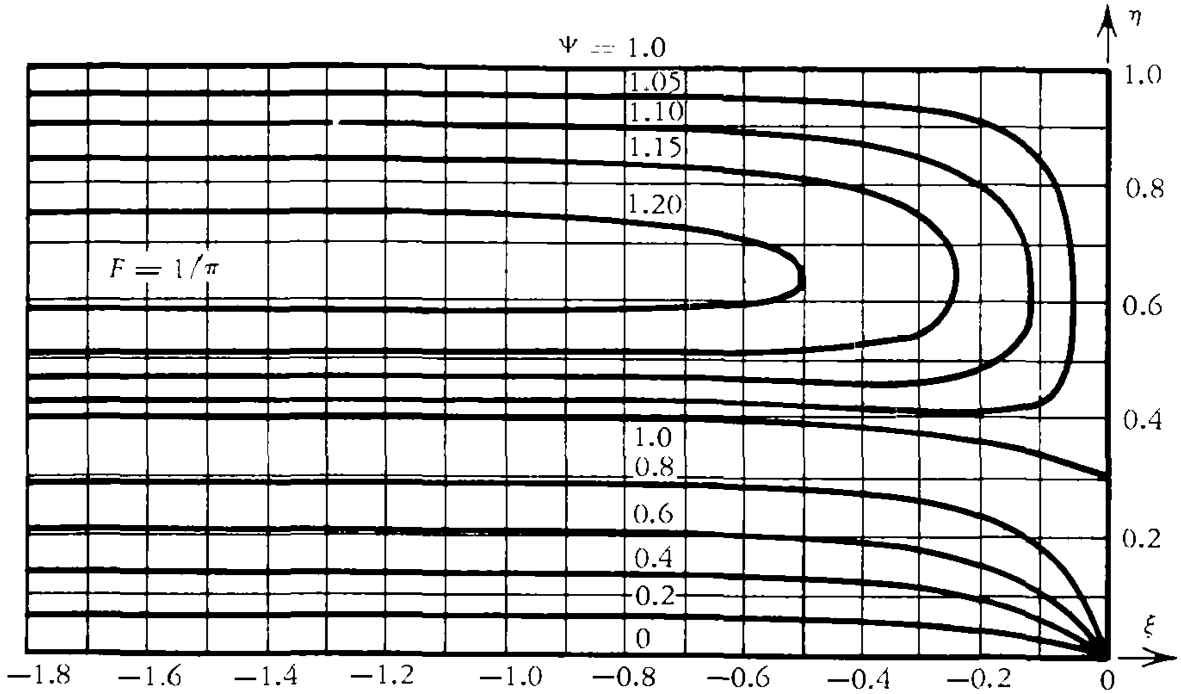


FIGURE 9 (*continued*). (c) $F = 0.35$. (d) $F = 0.32$.



(e)

FIGURE 9 (*continued*). (e) $F = 1/\pi$ (for which the assumed upstream condition is violated).

For F smaller than π^{-1}, an exact solution can still be obtained if waves are allowed upstream, and if $d\rho/d\psi'$ is still constant. The solution by separation of variables is

$$\Psi = \eta + \frac{2}{\pi}\left\{\sum_{n=1}^{N} \frac{\sin n\pi\eta}{n}\left[\cos (F^{-2} - n^2\pi^2)^{1/2}\xi + f(n)\sin (F^{-2} - n^2\pi^2)^{1/2}\xi\right]\right.$$
$$\left. + \sum_{n=N+1}^{\infty} \frac{\sin n\pi\eta}{n} \exp (n^2\pi^2 - F^{-2})^{1/2}\xi\right\}, \qquad (41)$$

if F is in the range

$$\frac{1}{(N+1)\pi} < F < \frac{1}{N\pi}.$$

The corresponding density far upstream is, since $d\rho/d\Psi$ is constant,

$$\rho = \rho_0 - (\rho_0 - \rho_1)\eta - \frac{2(\rho_0 - \rho_1)}{\pi}\sum_{n=1}^{N}\frac{\sin n\pi\eta}{n}$$
$$\times \left[\cos (F^{-2} - n^2\pi^2)^{1/2}\xi + f(n)\sin (F^{-2} - n^2\pi^2)^{1/2}\xi\right]. \qquad (42)$$

In (41) and (42), $f(n)$ is any function of n. Of course, if elaborate care is taken to insure that the upstream conditions are described by (41) and (42), there is no reason why the solution provided by them is unacceptable, at least outside of regions of closed streamlines. However, if the initial density variation with height is linear, and the discharge through the sink is small, it would be fortuitous indeed if the upstream density variation would adjust itself to the form (42). To test what would actually happen, Debler [1959] conducted a series of experiments for a range of Froude numbers, some of which above and many of

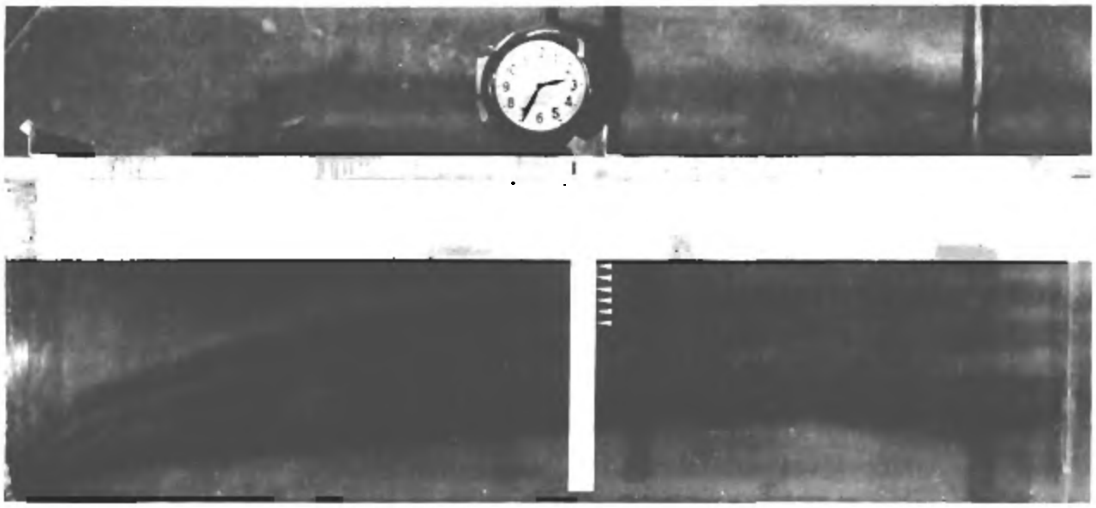


FIGURE 10. Pattern of two-dimensional stratified flow into a sink at $F = 0.35$, after Debler [1959]. (*Courtesy of the American Society of Civil Engineers.*)

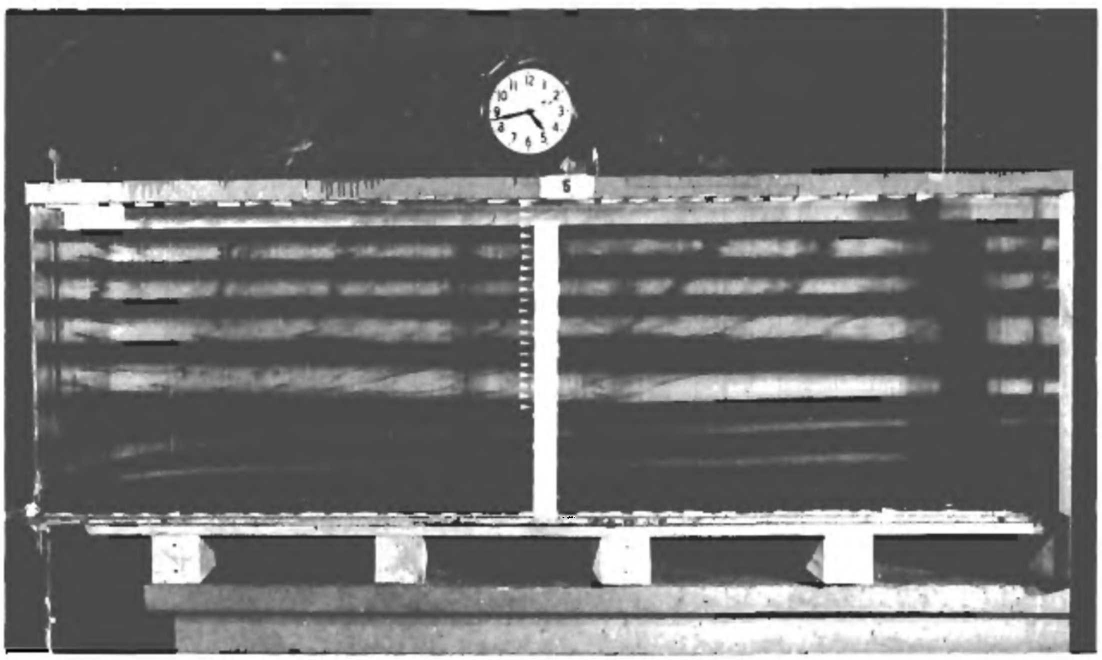


FIGURE 11. Pattern of two-dimensional stratified flow into a sink at a Froude number (based on d) much smaller than $1/\pi$, after Debler [1959]. (*Courtesy of the American Society of Civil Engineers.*)

which below π^{-1}. Figure 10 shows the flow pattern at $F = 0.35$ $(>\pi^{-1})$. The resemblance of this pattern to that obtained from calculation (Fig. 9c) is rather striking. For smaller F, the fluid is divided into a flowing region and an essentially stagnant region (Fig. 11). The Froude number F_1 based on the depth d_1 of the flowing region is, according to Debler's experiments, about 0.24, which is near but smaller than $1/\pi$. Now the Froude number based on the total depth d is, according to (30), (36), and (37),

$$F = \frac{1}{\sqrt{-A}} = \frac{U'}{d}\sqrt{\frac{\rho_0}{g\beta}}, \tag{43}$$

in which

$$\beta = -\frac{d\rho}{dz} \quad (\text{at} \quad \xi = -\infty) \quad = \frac{\rho_0 - \rho_1}{d}. \tag{44}$$

Let $q = U'd$, which is the volumetric discharge (per unit distance perpendicular to the plane of flow) based on the modified velocity U'. The Froude number based on d can be written as

$$F = \frac{q}{d^2}\sqrt{\frac{\rho_0}{g\beta}}. \tag{43a}$$

The Froude number F_1 based on d_1 used by Debler is

$$F_1 = \frac{q}{d_1^2}\sqrt{\frac{\rho_0}{g\beta}}. \tag{45}$$

Thus if F is much* greater than $1/\pi$, separation of one part of the fluid from the rest is impossible, but if F is much smaller than $1/\pi$, the fluid will separate, and the Froude number F_1 based on the depth d_1 of the flowing layer is of the order of $1/\pi$, and is, according to Debler's experiments, 0.24. It is significant that the difference between$1/\pi$ and 0.24 is not large compared with either of these numbers.

That at small values of F defined by (43a) the value of F_1 should remain constant is not fortuitous, but can be proved rigorously within the frame work of inviscid-fluid flow. Figure 12a indicates a flow at a small Froude number. The flow separates. Let there be a solution characterized by a flowing layer with an overlying stagnant layer, in which the density gradient is $-\beta$. On the free streamline AB, the boundary condition is that the pressure must be the hydrostatic pressure determined from the stagnant layer, namely (with ρ_A denoting the pressure at A)

$$p = p_A - \int_{d_1}^{z} g\rho\, dz = p_A + g\rho_A(d_1 - z), \tag{46}$$

since the density ρ_w in Fig. 19 can be assumed constant, so that

$$\rho_w = \rho_A .$$

Now the density along it on the flowing side is also ρ_A, the density at A. Thus the Bernoulli equation is

$$\frac{\rho_A}{2}(u^2 + v^2) + p + g\rho_A z = \text{constant},$$

* See Section 8.

or

$$\frac{\rho_0}{2}(u'^2 + v'^2) + p + g\rho_A z = \frac{\rho_0 U'^2}{2} + p_A + g\rho_A d_1, \tag{47}$$

in which u and v are the actual velocity components, u' and v' the modified velocity components, and U' is the value of u' at A. Combination of (46) and (47) yields

$$\rho_0(u^2 + v^2 - U'^2) = 0, \tag{48}$$

or

$$\left(\frac{\partial \Psi}{\partial \xi}\right)^2 + \left(\frac{\partial \Psi}{\partial \eta}\right)^2 = 1, \tag{48a}$$

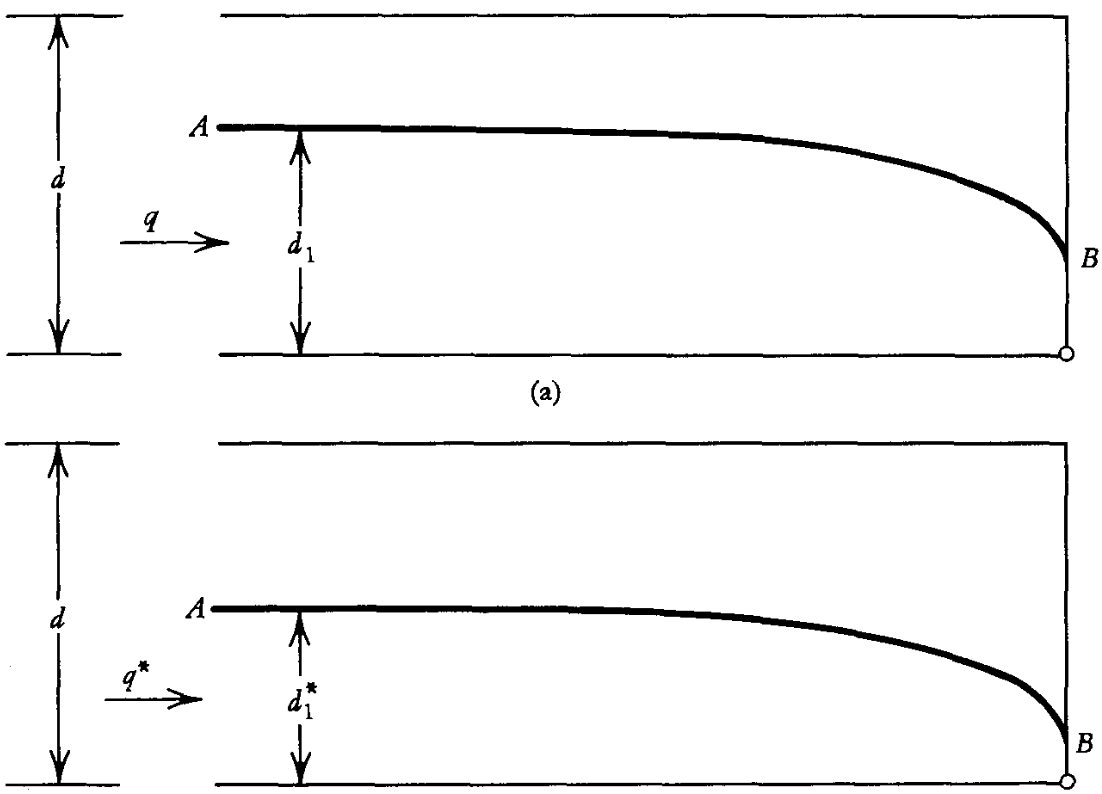


FIGURE 12. (a) Two-dimensional stratified flow into a sink with a stagnation zone, at a small Froude number. (b) Two-dimensional stratified flow into a sink with a stagnation zone, at a smaller Froude number.

if ξ and η are based on d_1, and $\Psi = \psi'/U'd_1$. The equation governing the flow in the moving layer is

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\Psi - F_1^{-2}\eta = -F_1^{-2}\Psi. \tag{49}$$

The kinematic boundary conditions on the rigid boundaries remain unchanged. Thus the solution of the governing differential system remains valid for any U' and d_1, provided F_1 remains constant. In other words, if at a smaller value (q^*, say) of q, the velocity determined by a solution is everywhere reduced by a factor d_1^*/d_1 (see Fig. 12b), and the linear scale of the flow by the same factor, the governing differential system remains satisfied. Thus the constancy of F_1 is based on the similarity of flow, which exists only if the sink has zero dimension and the density gradient is constant.

It is conceivable that in certain engineering processes the aim is to *prevent* separation (of fluid from fluid) at small rates of flow, or small F's. Debler's experiments indicate that when F is much less than $1/\pi$ something has to be done to achieve the elimination of a stagnation region. The work of Yih, O'Dell, and Debler [Yih *et al.*, 1962] indicate that this can be achieved by raising the bottom of the channel (by an amount varying with ξ), principally near the sink. If

$$\frac{1}{2\pi} < F < \frac{1}{\pi} \qquad \text{and} \qquad k\varepsilon = \pi,$$

one may impose the following modified boundary conditions at $\xi = 0$:

$$\Psi = 1 \qquad \text{for} \quad \varepsilon \leq \eta \leq 1,$$

$$\Psi = 1 - \frac{k^2}{\pi^3}\sin k\eta \qquad \text{for} \quad 0 < \eta \leq \varepsilon,$$

$$\Psi = 0 \qquad \text{at} \quad \eta = 0.$$

The boundary conditions at the plane boundaries are unchanged. Then since

$$\lim_{\varepsilon \to 0}\int_0^\varepsilon k^2 \sin k\eta \sin n\pi\eta \, d\eta = n\pi^2,$$

the solution by the method of separation of variables, for the limiting case of vanishing ε, is

$$\Psi = \eta - \frac{2}{\pi}\sum_{n=1}^{\infty}\left(n - \frac{1}{n}\right)\sin n\pi\eta \exp\,(n^2\pi^2 - F^{-2})^{1/2}\,\xi, \tag{50}$$

which is free from waves. The series and all its derivatives are convergent for any finite ξ (negative, of course), however small in numerical value. Therefore though the series itself and its derivatives are certainly not convergent for ξ equal to zero, their limits as $\xi \to 0$ nevertheless exist. The flow pattern (with $\Psi = \psi'/U'd$) represented by (50) is shown in Fig. 13a, for $F = 3/4\pi$, from which it is evident that a properly raised channel bottom near the sink is needed to draw the fluid from all levels.

For

$$\frac{1}{3\pi} < F < \frac{1}{2\pi},$$

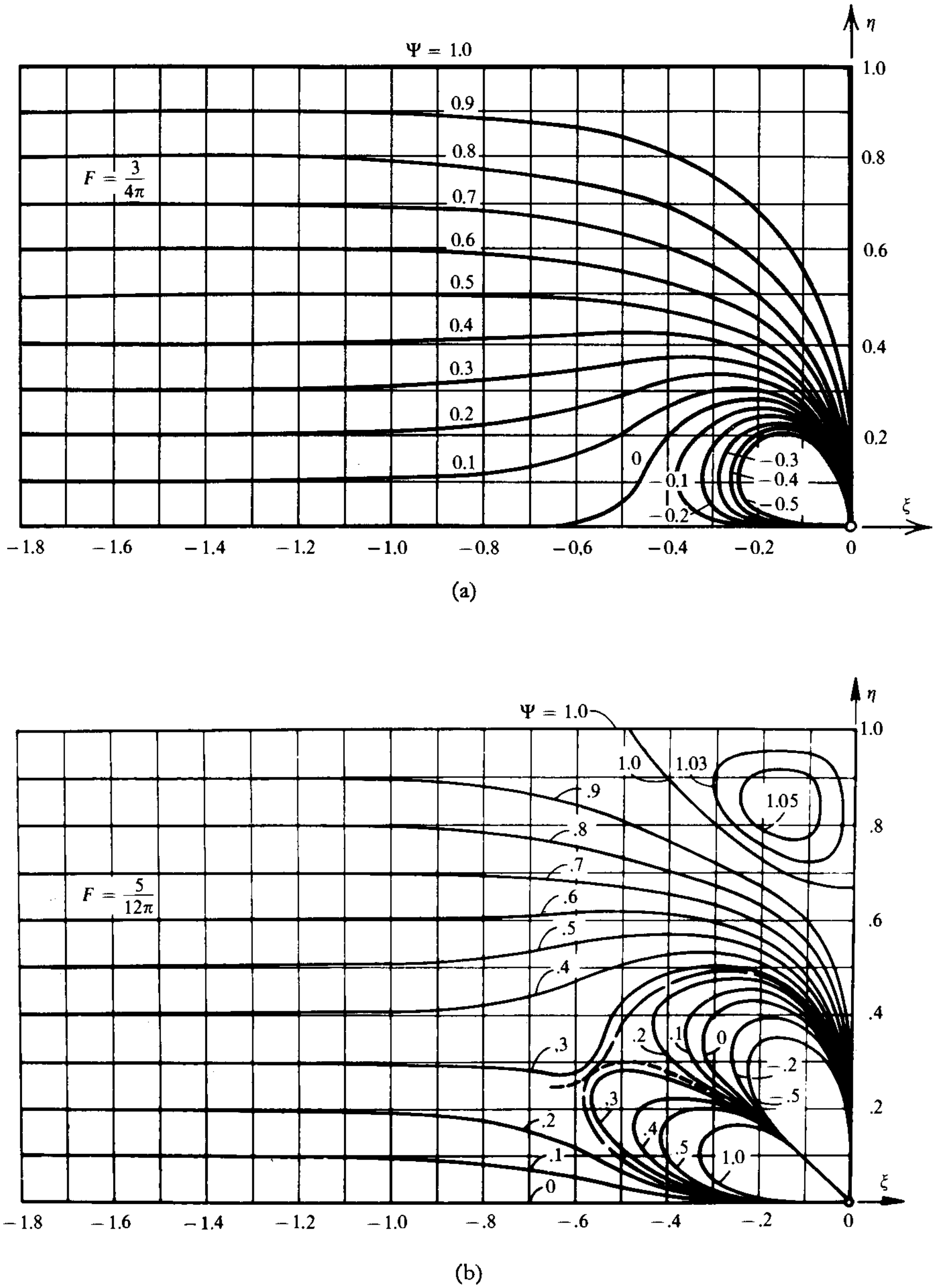


FIGURE 13. (a) Two-dimensional stratified flow into a sink superposed on a quadripole at $F = 3/4\pi$. The streamline $\psi = 0$ provides the form of a structure for preventing stagnation. (b) Two-dimensional stratified flow into a sink superposed on an octa-pole at $F = 5/12\pi$. The dashed line provides the form of a structure for preventing stagnation. (*Courtesy of the American Society of Mechanical Engineers.*)

the proper boundary condition for Ψ at $\xi = 0$ and $0 < \eta \leq \varepsilon$ is

$$\Psi = 1 - \frac{5}{4\pi^3} k^2 \sin k\eta + \frac{1}{3\pi^5} k^4(\sin k\eta + 2 \sin 2k\eta),$$

and the solution is

$$\Psi = \eta - \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{5n}{4} - \frac{n^3}{4} - \frac{1}{n}\right) \sin n\pi\eta \cdot \exp{(n^2\pi^2 - F^{-2})^{1/2}\xi}. \quad (51)$$

Again Ψ and its derivatives are convergent at any finite ξ, and possess limits as $\xi \to 0$. The flow pattern represented by (51) is shown in Fig. 13b, for $F = 5/12\pi$. It is interesting that an opening at $\eta = 0$ is now necessary,

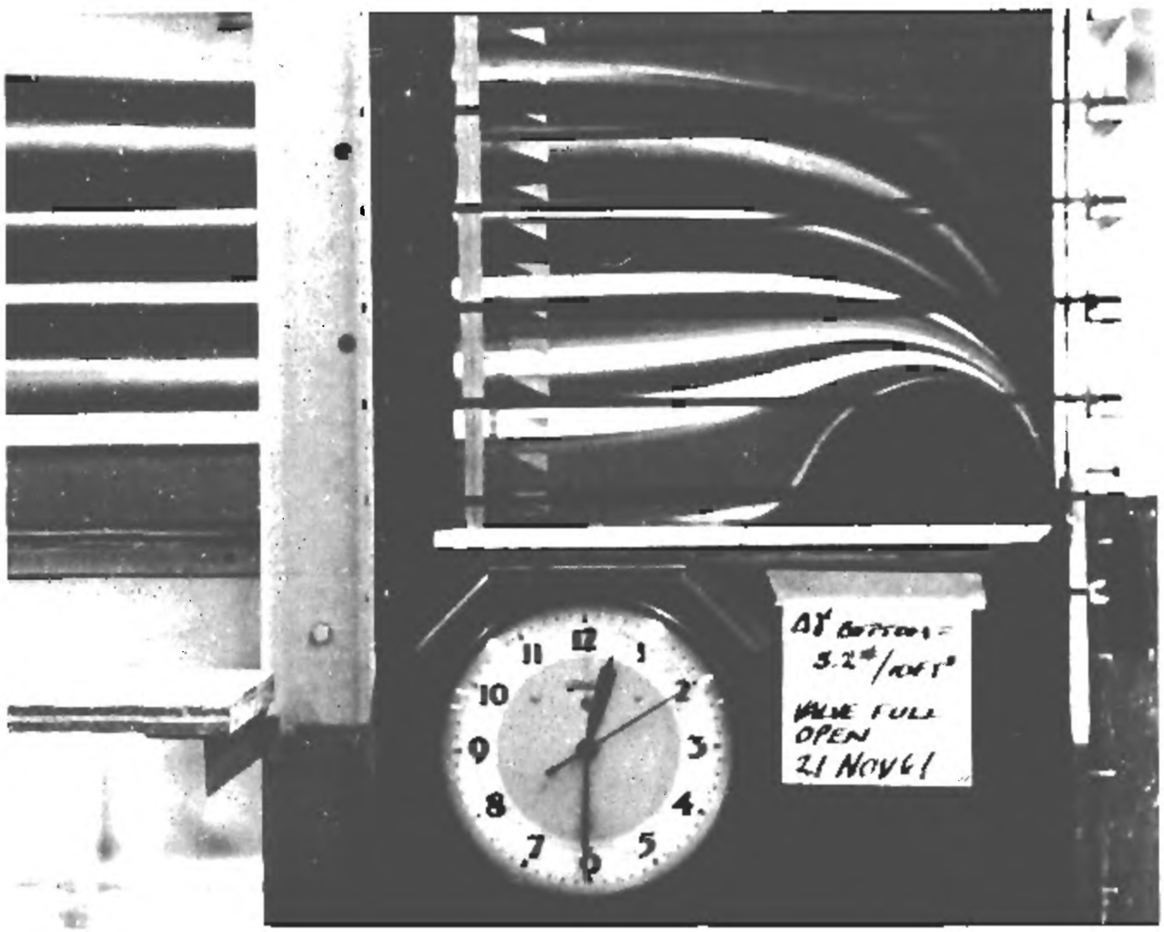


FIGURE 14. Actual flow pattern at $F = 3/4\pi$, with barrier constructed according to the streamline $\psi = 0$ in Fig. 13a. (*Courtesy of the American Society of Mechanical Engineers.*)

because the barrier is now so high that, whereas the upper fluid is greatly "encouraged" to come down, the lower fluid cannot be expected to climb over it. To prevent the lower part from being stagnant, it is necessary to provide a second access to the sink.

Experiments done by Debler [Yih *et al.*, 1962] have confirmed the theoretical predictions in the main. (See Figs. 14 and 15.) In general, a barrier

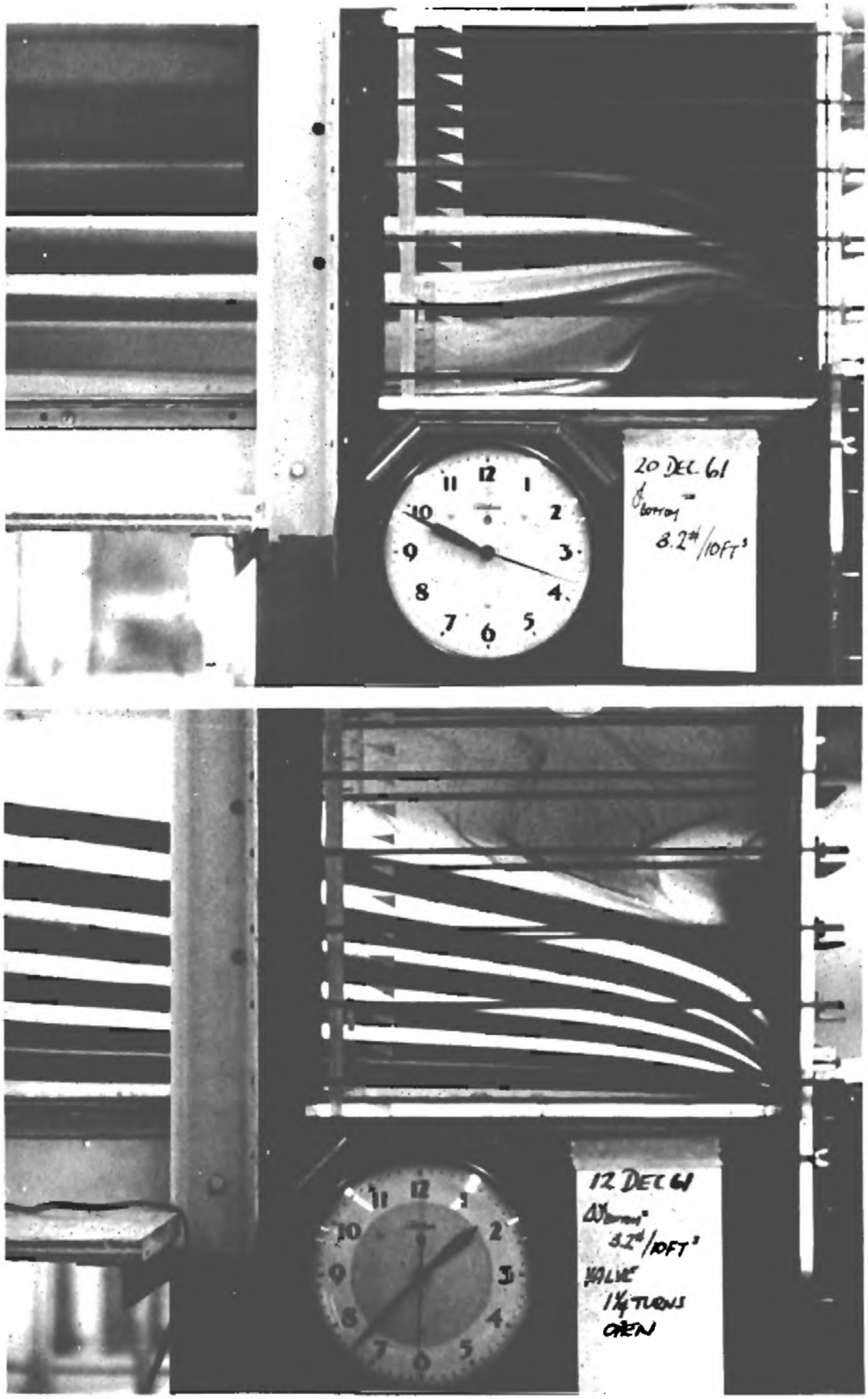


FIGURE 15. (Top) Actual flow pattern at $F = 0.159$ with the same barrier as in Fig. 14. (Bottom) Actual flow pattern at $F = 0.159$ without barrier. (*Courtesy of the American Society of Mechanical Engineers.*)

constructed for a particular low Froude number will prevent stagnation at other Froude numbers not too much lower than it, although small regions of stagnancy may exist, and the difference in flow pattern at low Froude numbers produced by a barrier placed near the sink is very striking indeed.

7. CLASS II (*continued*): STRATIFIED FLOW OVER A BARRIER

Two-dimensional flows of a stratified fluid over a barrier have close bearing on meteorology, because atmospheric flows over mountain ranges are such flows. As will be shown in a later section, the effect of static compressibility in an atmospheric flow can be properly accounted for by using the potential-density distribution of the atmosphere as the density distribution in an incompressible fluid simulating the atmosphere.

Linear theories dealing with atmospheric flows over a mountain range have been presented in Chapter 2. Large-amplitude flows have been treated mainly by Long [1955] and Yih [1960b]. Their methods of solution are different, and will be presented separately. Among the many questions to be answered are: (1) whether, for given upstream conditions, lee waves will occur; (2) if they do occur, what are the wavelengths and amplitudes of the various wave components. These questions are answered in the treatment of Long and Yih. Furthermore, Long's experiments [1955] clearly indicated that jets and regions of reverse flow become more and more numerous and pronounced as the modified Froude number is decreased or as the height of the barrier is increased. Although Long's theoretical calculations have also borne out this situation, a simple and direct explanation of it is still lacking. Such an explanation will be provided in this section.

Long, whose theoretical and experimental work on large-amplitude waves in a stratified fluid will be remembered as remarkable contributions to dynamic meteorology, considered a barrier extending over the range

$$-b \le \xi \le b,$$

with a profile approximately described by

$$\eta_s = f(\xi), \tag{52}$$

in which ξ and η are defined by (18), with d identified with the depth of the troposphere. Outside of the range of the barrier, the flow is assumed to be confined between two horizontal planes at distance d apart. The lower plane represents the flat part of the ground, and the upper plane, which extends all the way through $\xi = -\infty$ to $\xi = \infty$, simulates the tropopause, where any disturbance caused by ground topography is neglected. If the upstream condition is again described by (36) and (37), the governing equation is again (38), and Long's solution by separation of variables is

$$\left.\begin{aligned}
\Psi &= \eta + \sum_{N+1}^{\infty} A_n e^{a_n\xi} \sin n\pi\eta \qquad \text{for} \quad \xi \le -b, \\
\Psi &= \eta + \sum_{0}^{\infty} E_n \cos \frac{n\pi\xi}{b} \sin c_n(1 - \eta) \\
&\quad + \sum_{n=1}^{N} (F_n \cos a_n\xi + G_n \sin a_n\xi) \sin n\pi\eta \\
&\quad + \sum_{N+1}^{\infty} (H_n e^{a_n\xi} + M_n\, e^{-a_n\xi}) \sin n\pi\eta \qquad \text{for} \quad |\xi| \le b, \\
\Psi &= \eta + \sum_{n=1}^{N} (B_n \cos a_n\xi + C_n \sin a_n\xi) \sin n\pi\eta \\
&\quad + \sum_{N+1}^{\infty} D_n e^{-a_n\xi} \sin n\pi\eta \qquad \text{for} \quad \xi \ge b,
\end{aligned}\right\} \tag{53}$$

in which

$$\left.\begin{aligned}
a_n &= |F^{-2} - n^2\pi^2|^{1/2}, \\
E_n &= \frac{1}{b \sin c_n} \int_{-b}^{b} f(\xi) \cos \frac{n\pi\xi}{b}\, d\xi \qquad \text{for} \quad n \ge 1, \\
E_0 &= \frac{1}{2b \sin F^{-1}} \int_{-b}^{b} f(\xi)\, d\xi, \\
c_n &= \left(F^{-2} - \frac{n^2\pi^2}{b^2}\right)^{1/2},
\end{aligned}\right\} \tag{54}$$

and N is the last value of n which makes $F^{-2} - n^2\pi^2$ greater than zero. The requirement that there be no upstream waves is embodied in the first equation in (53). The constants A_n to M_n (except E_n, which are determined by the shape of the barrier) are determined by imposing the conditions that Ψ and $\partial\Psi/\partial\xi$ should be continuous at $\xi = -b$ and $\xi = b$. If Ψ is continuous along these two lines, all the derivatives of Ψ with respect to η (so long as they exist) are the same along the two lines whether they are approached from the right or the left. From (38) it can be seen that if Ψ and $\partial^2\Psi/\partial\eta^2$ are matched so is $\partial^2\Psi/\partial\xi^2$. By differentiating (38) it can be shown that all the higher derivatives are matched if $\partial\Psi/\partial\xi$ is matched. Therefore the two latter parts of the solution (54) are analytic continuations of the first. In matching the three parts of the solutions, it is sufficient to match the coefficients of the orthogonal functions $\sin n\pi\eta$. This involves the expansion of $\sin c_n(1 - \eta)$ into a Fourier series—an expansion which can always be obtained by quadratures.

To illustrate the actual calculation involved, consider the case (with Long)

$$f(\xi) = \frac{a}{2}\left(1 + \cos\frac{\pi\xi}{b}\right) \quad \text{for} \quad |\xi| \le b, \qquad f(\xi) = 0 \quad \text{elsewhere.} \tag{55}$$

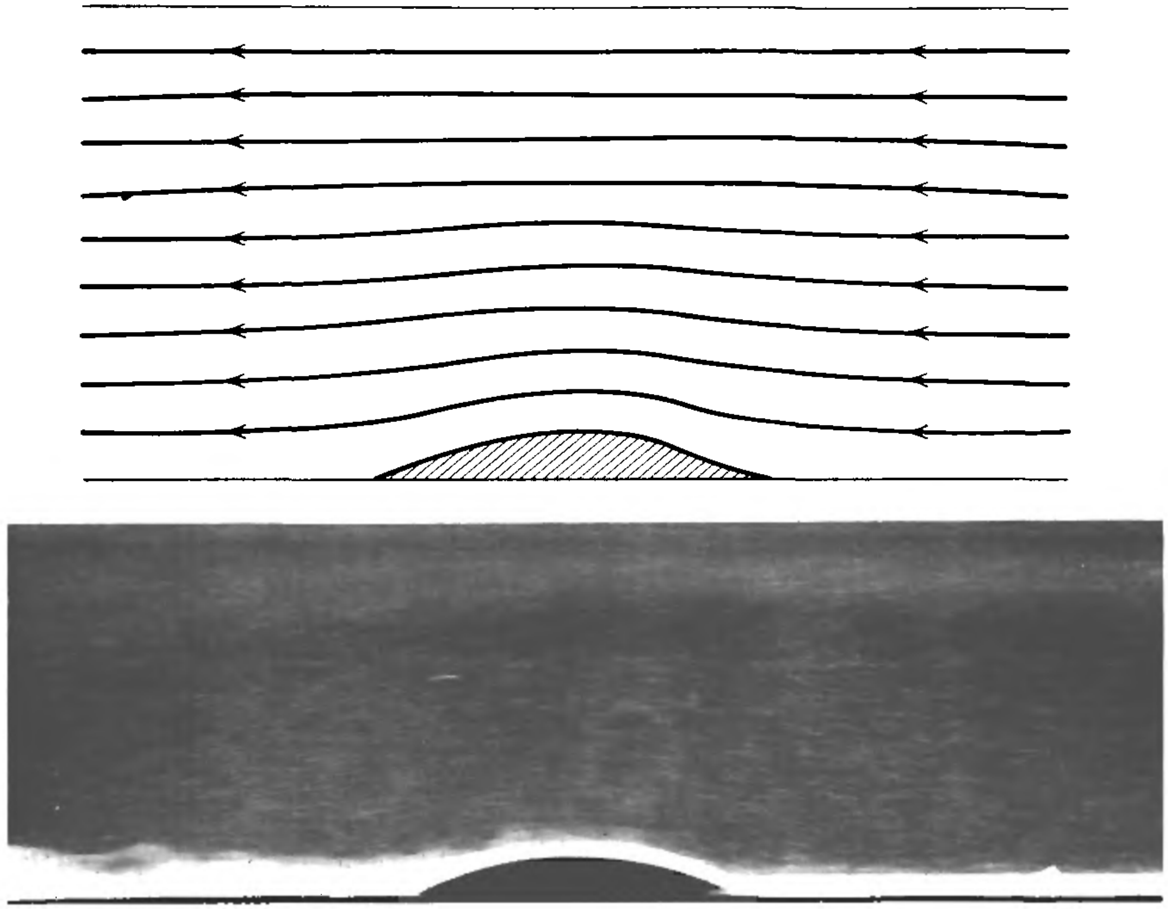


FIGURE 16a. Comparison of patterns of flow over a barrier obtained theoretically and experimentally by Long [1955]. (*Courtesy of Tellus.*) Theoretical: $F = 0.50$; $a/d = 0.10$; $b = 0.40$; $a = 0.14$. Experimental: $F = 0.35$; $a/d = 0.092$; $b = 0.42$.

In this case,

$$
\begin{aligned}
E_0 &= \frac{1}{2b \sin F^{-1}} \int_{-b}^{b} f(\xi)\, d\xi = \frac{a}{2 \sin F^{-1}}, \\
E_1 &= \frac{1}{b \sin c_1} \int_{-b}^{b} f(\xi) \cos \frac{\pi \xi}{b}\, d\xi = \frac{a}{2 \sin c_1}, \\
E_2 &= E_3 = \cdots = 0 .
\end{aligned} \tag{56}
$$

For $n > N$, the equations for matching Ψ and $\partial\Psi/\partial\xi$ at $\xi = -b$ are

$$A_n e^{-a_n b} - H_n e^{-a_n b} - M_n e^{a_n b} + \frac{a n\pi}{(n\pi)^2 - c_1^2} - \frac{a n \pi}{(n\pi)^2 - F^{-2}} = 0, \tag{57}$$

$$A_n e^{-a_n b} - H_n e^{-a_n b} - M_n e^{a_n b} = 0. \tag{58}$$

The corresponding equations for matchings at $\xi = b$ are

$$D_n e^{-a_n b} - H_n e^{a_n b} - M_n e^{-a_n b} + \frac{a n\pi}{(n\pi)^2 - c_1^2} - \frac{a n \pi}{(n\pi)^2 - F^{-2}} = 0, \tag{59}$$

$$-D_n e^{-a_n b} - H_n e^{a_n b} + M_n e^{-a_n b} = 0. \tag{60}$$

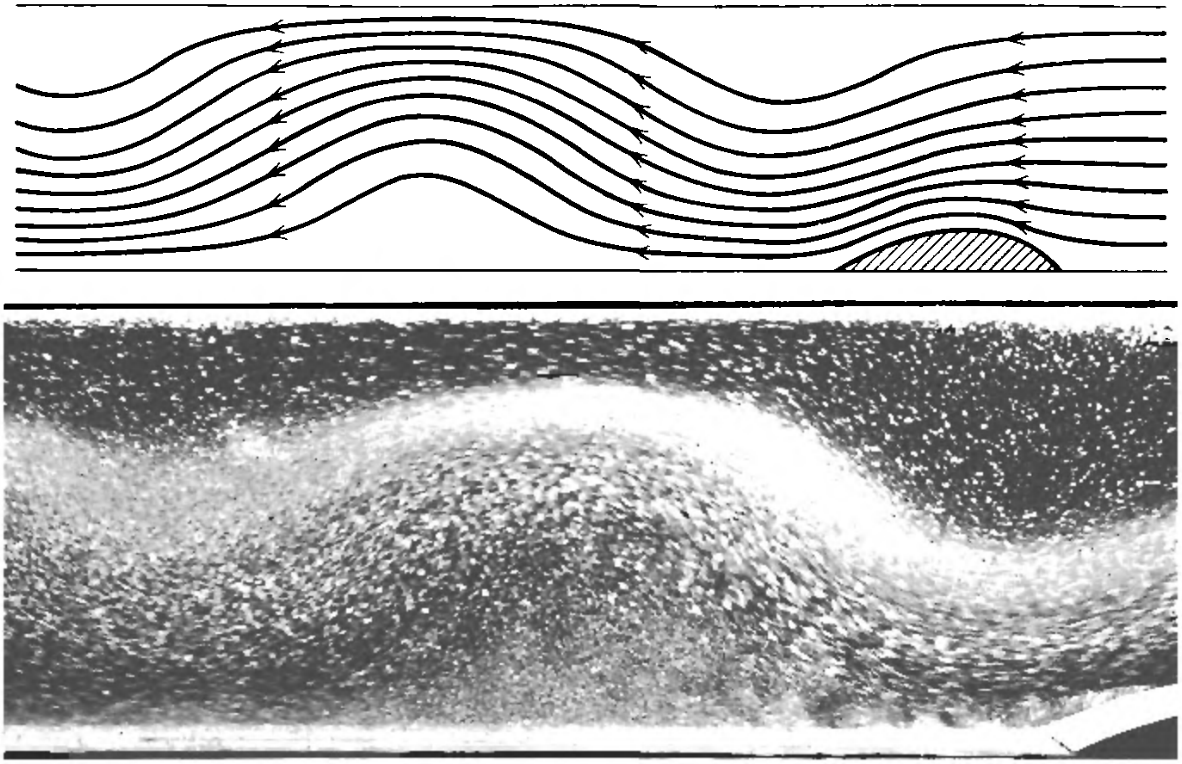


FIGURE 16b. Comparison of patterns of flow over a barrier obtained theoretically and experimentally by Long. Theoretical: $F = 0.25$; $a/d = 0.15$; $b = 0.40$; $a = 0.30$. Experimental: $F = 0.233$; $a/d = 0.151$; $b = 0.57$.

Solution of (57) to (60) yields

$$M_n = \frac{1}{2} e^{-a_n b} a n\pi \left(\frac{1}{(n\pi)^2 - c_1^2} - \frac{1}{(n\pi)^2 - F^{-2}} \right),$$

$$H_n = M_n, \qquad A_n = -M_n(e^{2a_n b} - 1), \qquad D_n = A_n.$$

For $n \leq N$, the determination of the constants is similar. Although the E's in (54) depend only on $f(\xi)$, it must not be construed that therefore (52) is an equation for a streamline, as one might do on letting η approach zero. The second and third series in the second of Eqs. (53) both contribute to Ψ, *even if* η_s *is infinitesimal.* If η_s is finite, not only will these two series destroy the qualification of the profile (52) as a streamline, but the first series must also be applied at finite η and not at $\eta = 0$. What Long's solution does accomplish, and accomplish very well, is the construction of an exact solution for an unknown profile *nearly* the same as that described by (52), so long as $\eta_s \ll 1$. The actual profile generally turns out to be unsymmetric, even if (52) is symmetric, but the skewness is slight. Figures 16a through f are a series of figures from Long's paper [1955] showing essential agreement of his theory with experiments. The agreement is indeed striking. However, when η_s is not small, Long's method is not applicable, in the sense that if one does apply it, the

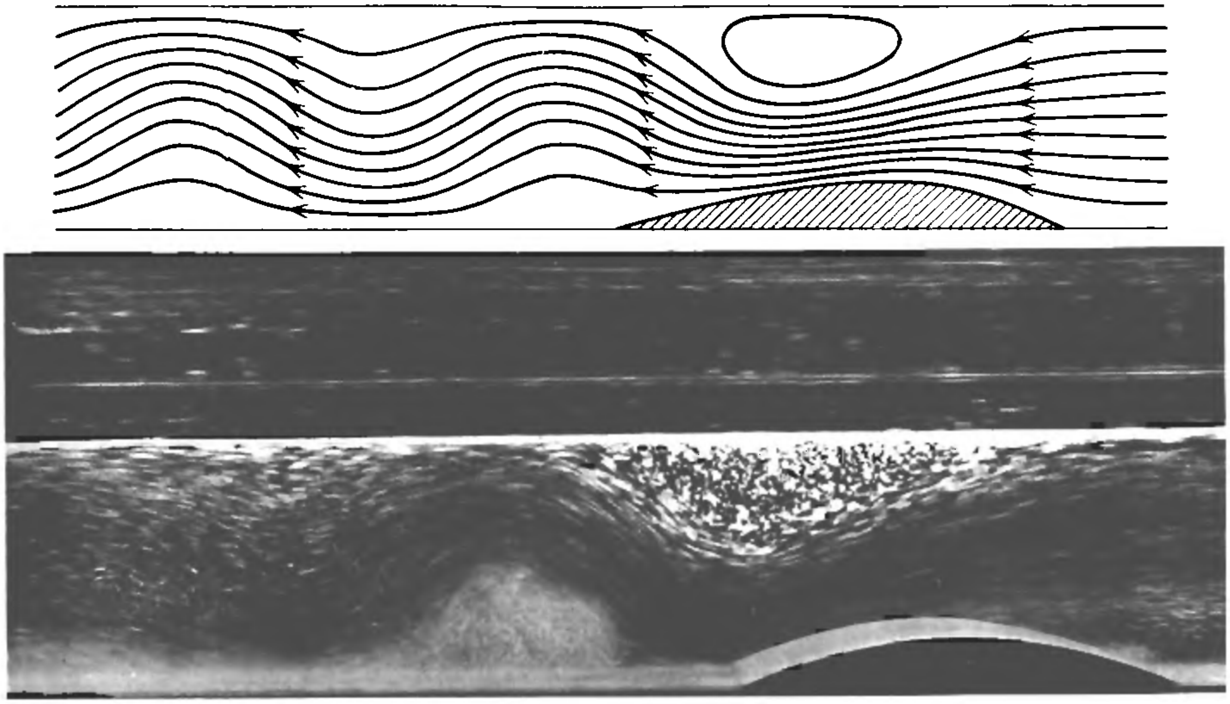


FIGURE 16c. Comparison of patterns of flow over a barrier obtained theoretically and experimentally by Long. Theoretical: $F = 0.20$; $a/d = 0.20$; $b = 1.0$; $a = 0.35$. Experimental: $F = 0.204$; $a/d = 0.20$; $b = 0.86$.

resulting profile of the barrier will bear little resemblance to the profile assumed.

Since $F^{-1} = c_0$, E_n $(n \geq 0)$ and E_0 become infinite when $\sin c_n = 0$, with $n_2 = 0, 1, 2, \ldots$. Therefore, for c_n equal to zero or an integral multiple of π, the formal procedure proposed by Long for constructing the solution breaks down. Examination of (54) reveals that the cases $c_n = 0$ do not introduce any serious difficulty, since the corresponding term in the E-series can be simply replaced by

$$\sum_0^\infty E_n \cos \frac{n\pi\xi}{b} \cdot (1 - \eta), \tag{61}$$

with E_n determined by

$$\begin{aligned} E_n &= \frac{1}{b} \int_{-b}^{b} f(\xi) \cos \frac{n\pi\xi}{b}\, d\xi \quad \text{for} \quad n \geq 1, \\ E_0 &= \frac{1}{2b} \int_{-b}^{b} f(\xi)\, d\xi. \end{aligned} \tag{62}$$

The E's so determined are still of the same order of magnitude as $f(\xi)$. The factor $(1 - \eta)$ in (61) is used to serve the same purpose as $\sin c_n(1 - \eta)$, and, like it, vanishes at $\eta = 1$, as is required by the boundary condition that $\eta = 1$ be a streamline.

An alternative inverse method for generating solutions for stratified flows over a barrier, due to Yih [1960b], is different from Long's in several respects. First, it is applicable to any barrier, not necessarily low. Second, it brings out

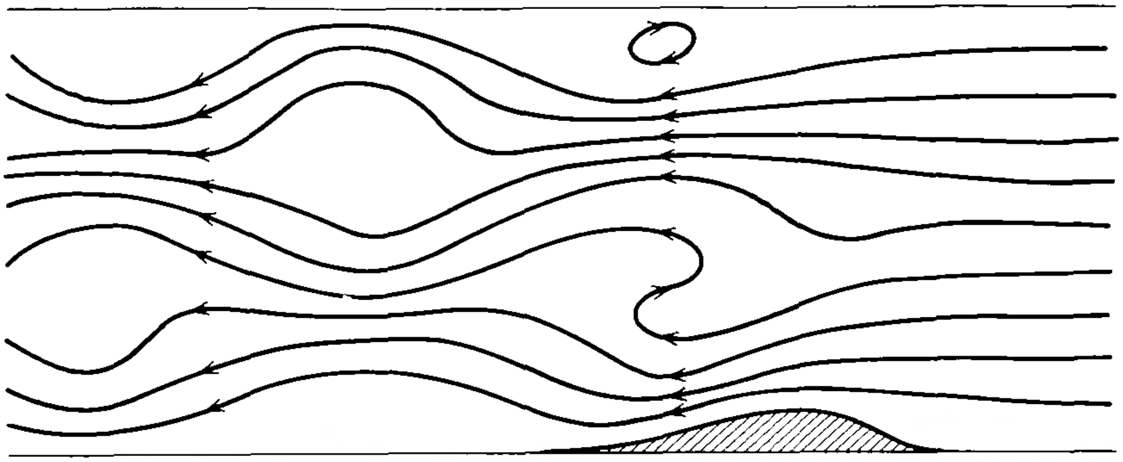


FIGURE 16d. Comparison of patterns of flow over a barrier obtained theoretically and experimentally by Long. Theoretical: $F = 0.091$; $a/d = 0.09$; $b = 0.4$; $a = 0.15$. Experimental: $F = 0.084$; $a/d = 0.085$; $b = 0.30$.

the lack of effect of the exact shape of the barrier on the amplitudes of the lee-wave components, which depend only on certain integral properties of the singularities generating the barrier. And third, it illustrates the generation of solutions of a partial differential equation by discontinuities, or singularities, much as potential flows are generated by sources, doublets, and vortices—in conjunction with uniform flows. The difficulty encountered in the application of Long's method when $c_n = p\pi$ (p = integer) does not arise in the alternative method to be presented here. However, Yih's method is an inverse method in that the barrier form is not specified *a priori* even approximately.

Yih's solution of (31) is

$$\left.\begin{aligned}\Psi_- &= \Psi_1 + \sum_{N+1}^{\infty} A_n e^{a_n\xi} \sin n\pi\eta \qquad \text{for} \quad \xi \le 0, \\ \Psi_+ &= \Psi_1 + \sum_{n=1}^{N} (B_n \cos a_n\xi + C_n \sin a_n\xi) \sin n\pi\eta \\ &\quad + \sum_{N+1}^{\infty} D_n e^{-a_n\xi} \sin n\pi\eta \qquad \text{for} \quad \xi > 0,\end{aligned}\right\} \tag{63}$$

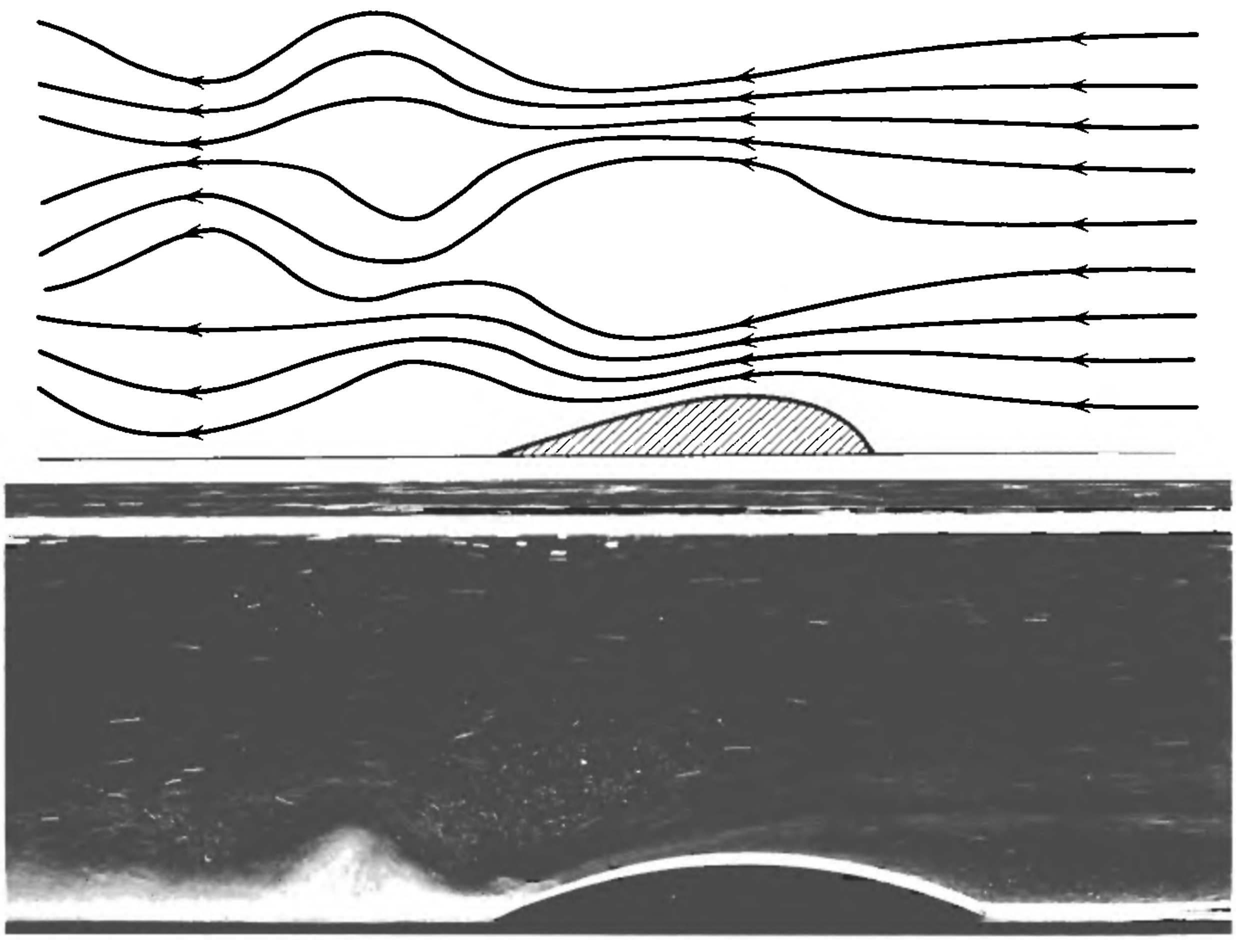


FIGURE 16e. Comparison of patterns of flow over a barrier obtained theoretically and experimentally by Long. Theoretical: $F = 0.083$; $a/d = 0.124$; $b = 0.4$; $a = 0.15$. Experimental: $F = 0.083$; $a/d = 0.138$, $b = 0.62$.

in which Ψ_1 describes the flow far upstream and is given by (33), N is an integer defined by

$$N^2\pi^2 < -B < (N+1)^2\pi^2,$$

and

$$a_n = |B + n^2\pi^2|^{1/2}.$$

If $-B = (p\pi)^2$ (p = integer), N is defined to be p. If $B = -F^{-2}$, then $\Psi_1 = \eta$, and $a_n = |F^{-2} - n^2\pi^2|^{1/2}$.

The coefficients A_n, B_n, C_n, and D_n are to be determined from the requirements that

$$\Psi_- = \Psi_+ \qquad \text{at} \quad \xi = 0, \tag{64}$$

$$\frac{\partial\Psi_-}{\partial\xi} - \frac{\partial\Psi_+}{\partial\xi} = f(\eta) \qquad \text{at} \quad \xi = 0, \tag{65}$$

in which

$$f(\eta) = 0 \qquad \text{for} \quad \eta \geq a,$$

and is arbitrary otherwise. The continuity of Ψ and of $\partial\Psi/\partial\xi$ at $\xi = 0$ and $a \leq \eta \leq 1$ and the differential equation (31) ensure the continuity of Ψ and of

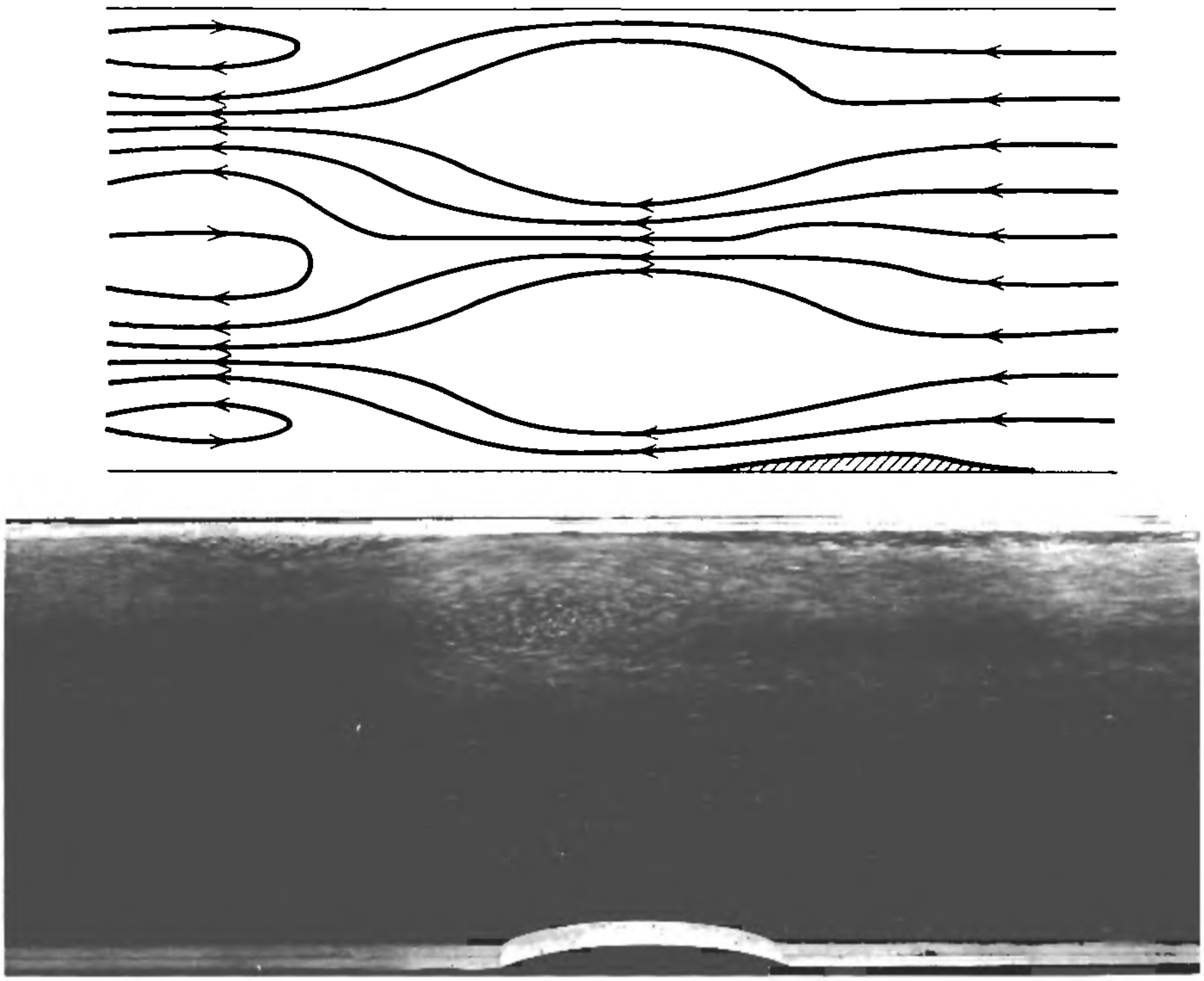


FIGURE 16f. Comparison of patterns of flow over a barrier obtained theoretically and experimentally by Long. Theoretical: $F = 0.077$; $a/d = 0.030$; $b = 0.4$; $a = 0.06$. Experimental: $F = 0.070$; $a/d = 0.056$; $b = 0.33$.

all its derivatives at $\xi = 0$, in the same range of η. The function $f(\eta)$ is a measure of the discontinuity in $\partial\Psi/\partial\xi$ for $0 < \eta < a$. Since $\partial\Psi/\partial\xi$ is proportional to the vertical velocity $v, f(\eta)$ corresponds to a vortex sheet of variable strength at $\xi = 0$, extending over the range $0 < \eta < a$. It implicitly determines the shape of the barrier, once the upstream conditions are given.

The condition (64) demands that

$$A_n = D_n \quad (n > N), \qquad \text{and} \qquad B_n = 0 \qquad \text{for} \quad n \leq N, \tag{66}$$

and (65) demands that

$$a_n(A_n + D_n) = 2\int_0^1 f(\eta)\sin n\pi\eta \, d\eta \qquad \text{for} \quad n > N, \tag{67}$$

$$a_n C_n = -2\int_0^1 f(\eta)\sin n\pi\eta \, d\eta \qquad \text{for} \quad n < N. \tag{68}$$

The coefficients A_n, C_n, and D_n can be determined from (66), (67), and (68).

Equation (68) is significant. It states that the amplitudes of the lee-wave components do not depend on the details of the function $f(\eta)$, but only on certain of its integral properties. Since it is the singularity function $f(\eta)$ that generates the shape of the barrier, it follows that these amplitudes do not depend on the details of the shape of the barrier, but on some of its integral properties manifested by integrals like the one in (68). Since there are infinitely many functions with the first N Fourier coefficients identical but with the rest of the Fourier coefficients different, there are infinitely many barriers which will create identical lee-wave components, provided the upstream conditions remain the same for all barriers. This result is a sort of St. Venant's principle in fluid flow.

For the case $A = B = -F^2$, $F = 3/4\pi$, and the vortex distribution

$$\begin{aligned} f(\eta) &= -10 \sin 5\pi\eta \qquad \text{for} \quad 0 < \eta < 0.2, \\ &= 0 \qquad\qquad\quad\ \ \text{for} \quad 0.2 < \eta < 1, \end{aligned}$$

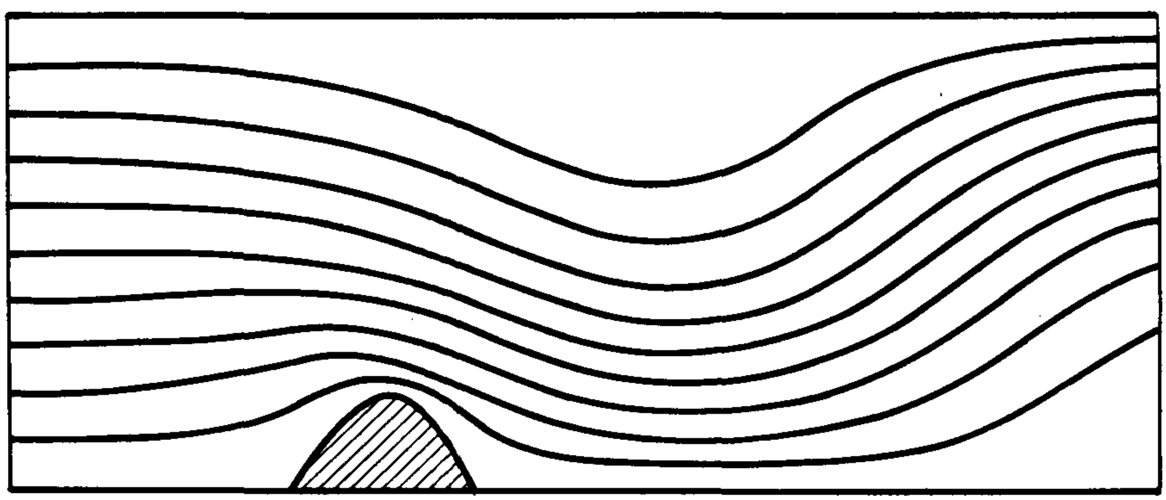


FIGURE 17. A stratified flow over a barrier with waves in the lee. $F = 3/4\pi$. (*J. Fluid Mech.*, *9*, *part 2. Courtesy of the Cambridge Univ. Press.*)

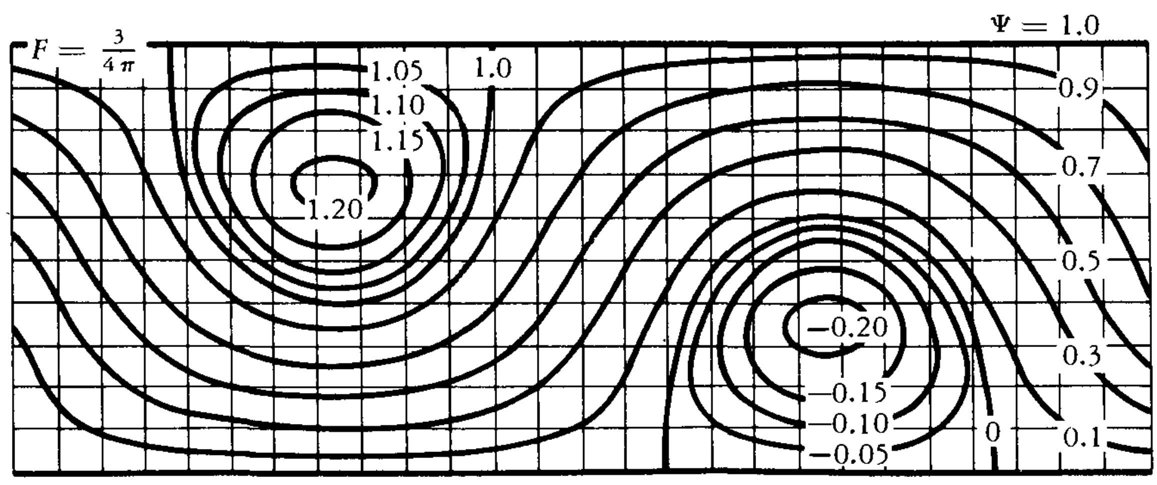


FIGURE 18. Details of the lee waves far downstream from a much higher barrier, for the same F. (*J. Fluid Mech.*, *9*, *part 2. Courtesy of the Cambridge Univ. Press.*)

the flow pattern is shown in Fig. 17. It has only one lee-wave component. Figure 18 shows the details of a stationary wave at the same Froude number ($3/4\pi$) but with a greater amplitude than that in Fig. 17. The wave pattern corresponds to

$$\Psi = \eta + \frac{2}{\pi}\cos(F^{-2} - \pi^2)^{1/2}\xi \cdot \sin n\pi\eta.$$

So far the singularities are assumed to be located at $\xi = 0$. The barrier created by such a line of singularities is likely to be blunt. To create elongated barriers, singularities at other locations may be used, say at $\xi = b$ and $\xi = c$, with singularity functions $f_1(\eta)$ and $f_2(\eta)$. The stream function is then

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4,$$

with Ψ_1 defined and Ψ_2 determined from $f(\eta)$ as before, but with Ψ_3 and Ψ_4 determined from $f_1(\eta)$ and $f_2(\eta)$ respectively. In Ψ_3 the variable ξ should be replaced by $\xi - b$, and in Ψ_4 it should be replaced by $\xi - c$. The form of Ψ_3 changes at $\xi = b$ and that of Ψ_4 changes at $\xi = c$, in the same fashion that the form of Ψ_2 changes at $\xi = 0$. For isolated vortices, $f(\eta)$ can be replaced by a Dirac function.

Instead of the conditions (64) and (65), conditions corresponding to a source distribution can be imposed. Thus, if we demand that

$$\Psi_- - \Psi_+ = f(\eta) \qquad \text{at} \quad \xi = 0, \tag{69}$$

$$\frac{\partial \Psi_-}{\partial \xi} = \frac{\partial \Psi_+}{\partial \xi} \qquad \text{at} \quad \xi = 0, \tag{70}$$

with $f(\eta) = 0$ for $\eta \geq a$, but otherwise arbitrary, the coefficients in (63) are determined from

$$A_n = -D_n \quad \text{for} \quad n > N, \qquad C_n = 0 \quad \text{for} \quad n \leq N, \tag{71}$$

$$A_n - D_n = 2\int_0^1 f(\eta)\sin n\pi\eta \, d\eta \qquad \text{for} \quad n > N, \tag{72}$$

$$B_n = -2\int_0^1 f(\eta)\sin n\pi\eta \, d\eta \qquad \text{for} \quad n \leq N. \tag{73}$$

If there are more than one line of sources, the generalization is similar to that given in the last paragraph. For a closed body, however, the total algebraic sum of the sources must be zero. That is to say,

$$\int_0^a f(\eta)\, d\eta + \int_0^{a_1} f_1(\eta)\, d\eta + \int_0^{a_2} f_2(\eta)\, d\eta = 0,$$

in which a, a_1, and a_2 are the extents of the singularity functions $f(\eta)$, $f_1(\eta)$, and $f_2(\eta)$, respectively.

If $f(\eta)$ is a Dirac function corresponding to a source of strength m at $\xi = -b$ and $\eta = h_s$, the nth ($n \leq N$) lee-wave component is represented by

$$-2m \sin n\pi h_s \cos a_n(\xi + b) \sin n\pi\eta.$$

If, in addition, there is a sink of the same strength at $\xi = b$ and $\eta = h_s$, the nth lee-wave component is

$$-2m \sin n\pi h_s [\cos a_n(\xi + b) - \cos a_n(\xi - b)] \sin n\pi\eta$$
$$= 4m \sin n\pi h_s \sin a_n b \sin a_n \xi \sin n\pi\eta.$$

If b is small, the amplitudes of the lee waves are proportional to $2mb$, which is the negative of the moment of the source and the sink. If $2mb$ is denoted by μ and if $2mb$ is kept constant as $b \to 0$, a doublet is obtained. Thus μ is the strength of a doublet with its axis directed upstream. The amplitude of the nth lee wave for a doublet situated at $\xi = 0$ and $\eta = h_s$ is

$$2\mu a_n \sin n\pi h_s.$$

For a doublet distribution from $\eta = 0$ to $\eta = a$, the amplitude of the nth lee wave is

$$2a_n \int_0^a \mu(\eta) \sin n\pi\eta \, d\eta.$$

If the doublet or doublet distribution is situated at $\xi = b$, the phase (in ξ) of the pertaining lee waves will be changed by $-b$.

Returning to the questions posed at the beginning of this section, we see that (if B is identified with $-F^{-2}$ in Yih's solution)

(1) lee waves will occur as soon as F is less than $1/\pi$, and there are N lee-wave components if

$$\frac{1}{(N + 1)\pi} < F < \frac{1}{N\pi},$$

(2) the wavelengths are

$$\lambda_n = \frac{2\pi d}{a_n},$$

and the amplitudes are given explicitly by both methods for an implicitly prescribed barrier and given upstream conditions.

For Yih's solution, similar conclusions are reached if $B \neq -F^{-2}$. It remains to explain why jets and regions of reverse flow become more and more numerous and pronounced as F decreases or as the barrier height increases (which is to say, as $f(\eta)$ increases). Aside from the term η in (53) and Ψ_1 in (63), which becomes η if $B = -F^{-2}$, all the other terms in (53) or in (63) represent collectively the disturbance to the flow due to the barrier. Now these terms correspond to a rather complicated flow pattern by themselves, rich in jets and reverse flows, especially if F is very small, so that there are many lee-wave components with large wave numbers a_n, which do not die out as $\xi \to \infty$. If $f(\eta)$ and its range a are large, the barrier will be high and extensive, and the terms representing barrier disturbance will be large. If they are large, the jets and reverse flow regions will not be easily "washed out" by the primary flow represented by η or Ψ_1, and will actually be seen in the flow—especially at small F, for which the disturbance flow abounds in jets and reverse-flow regions.

The constancy and magnitude of the modified Froude number F for *eddy regions* in a stratified flow offer a field for conjectures, of which there are some in the literature. Since the arguments intended to substantiate these conjectures seem rather unconvincing and sometimes purely assertive to this writer, they will not be presented.

8. CLASS II (*concluded*): THE PHENOMENON OF BLOCKING

The experiments of Long [1955], Debler [1959], and Yih [1959d] all indicate that at very small (modified) Froude numbers stagnation zones occur upstream and sometimes even downstream from a barrier or a moving body. This phenomenon is called blocking. The discussion given in Section 6 already indicated what kind of a solution is to be expected for stratified flow into a sink, when the Froude number is low. This flow will be discussed first. Blocking in a flow over a barrier will be treated next.

For stratified flow at Froude numbers less than $1/\pi$, the solution given by Yih [1958] cannot be utilized. A solution for low Froude numbers must provide for the possibility of the occurrence of a stagnation layer. Here one is confronted with the dilemma: On the one hand one wishes to construct a solution at low Froude numbers, and, on the other, the procedure in Yih's solution does not, without additional measures, produce a solution with parallel flow at infinity for low Froude numbers. The resolution of this dilemma is provided by the technique of allowing a fictitious sink distribution (Fig. 19a) from $\eta = a$ to $\eta = 1$ at $\xi = 0$, in such a way that the Froude number based on the total sum of the distributed sinks and the sink at the origin is greater than $1/\pi$, so that Yih's solution can be applied to the entire channel. If the fictitious sink distribution ends at $\eta = a$, the dividing streamline will be tangent to the line $\xi = 0$ at $\eta = a$, and will divide the flow into the sink at the origin and the flow into the fictitious sink distribution. At $\xi = -\infty$, all streamlines will be horizontal. The velocity and hence the pressure along the dividing streamline can then be computed. The pressure distribution corresponds to a hydrostatic pressure distribution resulting from *some* density distribution in the stagnant layer. If the pressure distribution along the dividing streamline corresponds to a stable density distribution in the stagnant layer, the solution obtained is an acceptable one for that density distribution. To obtain the solution for a *given* density distribution in the stagnant layer, the fictitious sinks and their range (from $\eta = a$ to $\eta = 1$) must be varied until the solution gives a pressure distribution along the dividing streamline corresponding to the given density stratification in the stagnant layer. The process is admittedly a laborious one, but it is the only one that can produce a solution involving a vortex sheet in the fluid. The flow into the fictitious sinks will be

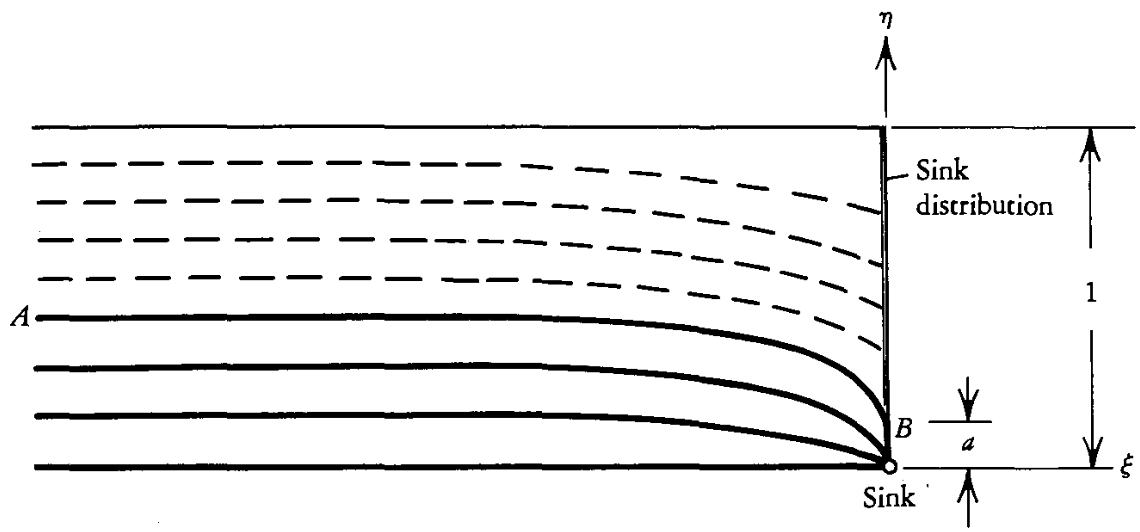


FIGURE 19. Schematic drawing for constructing a flow with a stagnation region. The stagnation region is occupied here by a fictitious flow into a fictitious sink distribution.

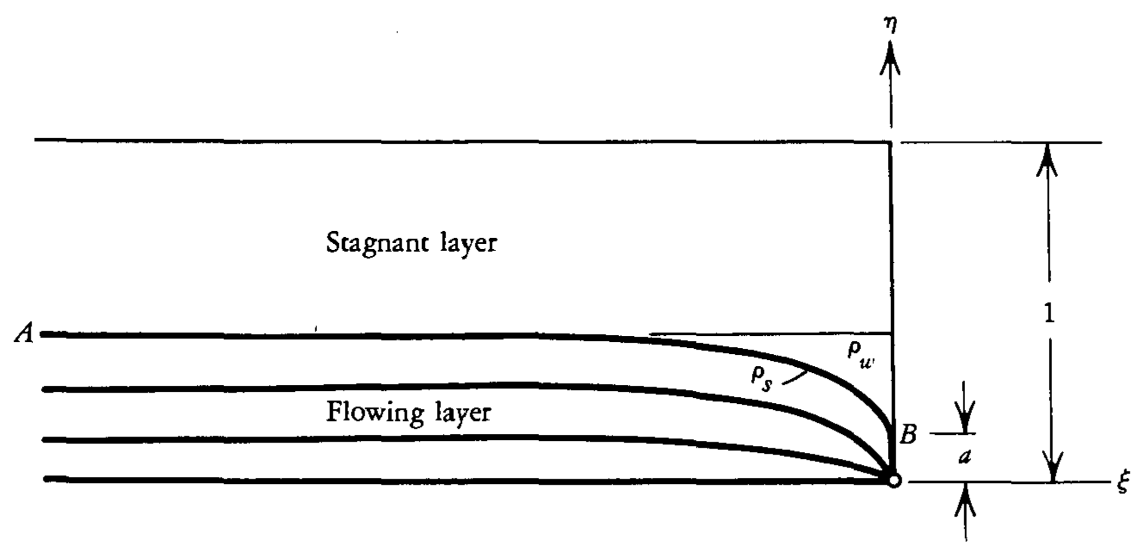


FIGURE 20. Model for actual two-dimensional flow into a sink at small Froude numbers (based on actual total depth d).

replaced by the stagnant layer after the solution has been obtained. This procedure was suggested to T. W. Kao, whose doctoral thesis treats the phenomenon of blocking. Kao [1963] solved two cases in which the density ρ_w in the wedge region in Fig. 20 is a constant rather than a linear function of z. The argument for assuming it to be constant is that the original linear stratification in the fluid can hardly be expected to be exactly preserved in the wedge region, because of the mixing that is constantly occurring there in reality. Since there is no way to determine ρ_w, Kao used the inverse method to find a flow and a ρ_w which satisfy the pressure condition on the dividing streamline, on which the density ρ_s is determined from the upstream condition. He considered two values of ρ_w and obtained for these two cases $F = 0.345$, where F is based on the total depth d in Fig. 12a, corresponding to an F_1 $(= U'/(g'd_1)^{1/2})$ of 0.675 in one case, and an even higher value of F_1 for the other case. It seems that solutions for a range of F_1 are possible. A discussion of this point, using a homogeneous fluid as an example, was given by Yih [1969b], who remarked that a separated flow of a homogeneous inviscid fluid into a sink, with the same geometry as in Figs. 19 and 20, is possible for whatever value of d_1, by the use of the Schwarz–Christoffel transformation, which Kao [1970] later did use to obtain such a flow. This solution would, of course, violate the persistence of irrotationality and would therefore be unacceptable. But its existence points to the possibility of having different values of F_1 for the solution of Kao's problem. Kao [1970], indeed, concluded from his new calculations that the range of F_1 is

$$0.33 \leq F_1 \leq \infty,$$

and gave the minimum value 0.33 for the unique value of F_1 for separated flows of a linearly stratified fluid into a sink, with the geometry shown in Fig. 20, and with $\rho_w = \rho_s$.

Later Kao [1976] reconsidered the selective-withdrawal problem, this time for stratified inviscid fluids with arbitrary stratification and even possibly with a lateral contraction. He mentioned, somewhat vaguely, an "initial velocity" q "induced" by a sink or whatever is causing the fluid to flow. This velocity seems to be nothing but the discharge divided by the cross section far upstream, with no regard to the possibility of velocity variation in this cross section. Then, if this q is greater than the greatest internal-wave velocity c_1 (of infinite or very long wavelength and of the first mode, of course), he asserts that no selective withdrawal is possible, whereas if q is less c_1, at least partial selective withdrawal is possible. When there is a lateral contraction q is taken at the point of narrowest cross section, but c_1 seems to continue to be the largest internal-wave velocity in the *uncontracted* channel. There is some ambiguity as well as weakness of the assertion here, since the density distribution at the contraction is not the same as in the uncontracted part when the fluid is flowing. He cited existing experimental results to support his criterion, which, applied perceptively by him to the cases cited, did seem to give plausible and credible results, and this criterion does have the merit of simplicity and generality.

And yet the criterion has not been reached through a rigorous theory, and even its application requires caution—because it is not unambiguous. For one thing, the location of the sink is only vaguely mentioned. Take, for instance, the case of two superposed layers of depth d_1 and d_2, between two rigid horizontal boundaries. Let the density ρ_1 in the upper layer be smaller than the density ρ_2 in the lower layer, and let d_1 be very much smaller than d_2, so that $d_1/d_2 \ll 1$. Then c_1 (in this case the only mode) is then very small, of the order of $[gd_1(\rho_2 - \rho_1)/(\rho_1 + \rho_2)]^{1/2}$. Then if the sink is located in the upper layer a very small q would make selective withdrawal impossible—not an improbable situation. But what if the sink is in the lower layer? The criterion given would again conclude the impossibility of selective withdrawal. But would this be right? I think for sufficiently small q, which can be larger than c_1, selective withdrawal would be possible. In other words, where the sink is located is of crucial importance. Indeed, if the sink is located at the interface, no q, however small, can give any discharge ratio other than

$$\frac{Q_1}{Q_2} = \frac{d_1}{d_2}\sqrt{\frac{\rho_2}{\rho_1}},$$

for gravity would then have no effect whatsoever, and internal gravity waves are entirely irrelevant. The so-called columnar disturbances, further considered by Kao in an investigation of the establishment of stratified flows, are nothing but very long waves.

For a weakly and linearly stratified fluid the location of the sink would not much affect the criterion so far as the *impossibility* of selective withdrawal is concerned. But the velocity distribution in the fluid would, at low Froude numbers, very much depend on the location of the sink. This dependence has been ignored in Kao's paper, which does not say how the velocity distribution is to be found in the general case. The distribution of velocity for internal waves of long wavelength is *not* the velocity distribution caused by the sink. But Kao seems to make no distinction between the two.

In dealing with unsteady flows, the acceleration of the fluid during establishment of steady flows has not been carefully considered by Kao. Perhaps he assumed tacitly a very slow establishment.

In spite of the above criticisms, Kao's criterion, with its simplicity and (somewhat insouciant) generality, is very useful as a practical rule in hydraulic engineering if it is applied with caution and intelligence.

There are many other works on the problem of selective withdrawal. Some of these [Koh, 1966; Wood, 1968; Lai and Wood, 1975] will be discussed in the notes at the end of this chapter. We now turn to a discussion of an important paper by Pao and Kao [1974].

Pao and Kao [1974] considered the establishment of the two-dimensional flow of a stratified fluid between two rigid horizontal boundaries into a line sink located in the lower boundary. In a first calculation effects of viscosity and diffusivity were neglected. Then these were taken into account, except at the rigid boundaries, where the nonslip condition was not imposed, presumably on the assumption that the boundary layers are so thin that their presence can be neglected. The flow patterns shown for (internal) Froude number equal to 0.20, a properly defined Reynolds number equal to 10^6, and a Schmidt number (ratio of kinematic viscosity to kinematic diffusivity) of 833 at a finite time and at the steady state (infinite time) give reasonable velocity profiles at some value of x (horizontal distance from the sink). But in these patterns the streamlines intersect the lower boundary, in violation of the condition that the lower boundary should be a streamline up to the locatión of the sink. The isopycnic lines corresponding to the flow patterns also intersect the lower boundary, violating the condition that the lower boundary should be an isopycnic line. It is not clear whether these violations were the result of some mistakes in calculation or of the inavoidable errors accumulated in the course of the computation.

Furthermore, the figures shown in their paper do not indicate a streamline dividing the stagnant zone from the flowing region, even though one velocity profile for the steady state does indicate the existence of a stagnant zone. What goes on in the stagnant zone was not described, graphically or otherwise, in their paper. It seems that since viscosity and diffusivity were both taken into account to produce the flow patterns, a recirculating region (an elongated eddy) is the only possible flow pattern in what was tacitly assumed to be a stagnant zone.

In any case, Pao and Kao [1974] did not attempt to give a critical Froude number below which the fluid at the upper boundary far upstream does not flow into the sink. For the inviscid and nondiffusive case, presumably, we are still to rely on the work of Kao [1970] to provide that critical Froude number (about 0.33). For a viscous fluid, whatever its diffusivity, the flow at infinity is always horizontal, with the well known parabolic velocity distribution, if the nonslip condition is imposed at both horizontal boundaries, as it should be, so that the discharge of the fluid between any two levels at infinity is already determined far upstream and continues unchanged toward the sink, even though the salinity of the fluid may change along the way, on account of the diffusivity. This change is not very significant, since in practice the salinity at the two horizontal boundaries is not maintained constant in the first place.

The problem formulated by Pao and Kao [1974], then, is well posed only within the confines of the validity of the following assumptions:

1. The channel is sufficiently long to allow differentiation of long waves (columnar disturbances) from shorter waves, but not so long as to allow the viscous boundary layers at the horizontal boundaries to grow to thicknesses of the same order as the total depth.
2. At the far end, therefore, is a reservoir from which the stratified flow issues. The effect of the contraction is negligible.
3. The Reynolds number is sufficiently high to allow the nonslip condition to be relaxed at the boundaries, given the length of the channel, and the Péclet number (equal to the Reynolds number times the Schmidt number) is sufficiently high to allow the imposition of the condition of

constant salinity at either horizontal boundary to be of no great importance (i.e., to be innocuous).

Within these confines, if the results are sufficiently accurate, they are meaningful and useful. The velocity distributions calculated by Pao and Kao are admirably verified by the experiments of Kao, Pao, and Wei [1974]. Evidently the assumptions listed above were satisfied for their measurements.

One final criticism of the work of Pao and Kao [1974] is that they assumed the initial motion of the fluid, when suddenly started, to be irrotational. This is in fact erroneous, for if the density is not uniform, vorticity will be created immediately as the motion is created, even if the effect of gravity is neglected.

In their discussion of the work of Yih [1958], Pao and Kao stated that "indeed in Yih's inviscid steady-state formulation the solution does not exist for $F = 1/\pi$, implying perhaps some abrupt changes near $F = 1/\pi$." There is no such implication, in spite of their inference. The very fact that Debler's experiments indicate constancy of the Froude number based on the depth of the flowing layer means that the depth of the flowing layer will gradually decrease as the Froude number decreases.

It is also interesting that if, when viscosity and diffusivity are taken into account, there is a very slowly flowing region at sufficiently low Froude numbers, there *must be* an elongated eddy occupying the stagnant corner (but not extending to infinity far upstream with a finite thickness), created by viscous traction and the requirement of continuity. This question has not been settled by Pao and Kao.

In spite of the criticism raised, it must be emphasized that Pao and Kao have made a valiant attempt at determining stratified flow by time establishment and pointed the direction for future work on this problem. The velocity distributions they provided, though somewhat in error, were previously unavailable.

Apart from the problem of selective withdrawal, there has been also a lot of interest in and discussion of the phenomenon of blocking in stratified flows over barriers, and there has been a good deal of confusion about what constitutes blocking. Many workers in the field and the related fields of rotating fluids and magnetohydrodynamics consider the phenomenon of blocking identical with the phenomenon of upstream wakes and as a consequence consider blocking unavoidable as soon as the flow becomes so slow as to permit waves to propagate upstream.

For clarity we shall define blocking as the phenomenon of stagnation of a layer of the fluid leading from an obstacle upstream to infinity. So defined, blocking or the lack of blocking is not synonymous with the presence or absence of the upstream influence of an obstacle. To see this in the simplest way, consider a layer of *homogeneous* liquid of depth d flowing supercritically in a straight channel with a level bottom at a speed U greater than $\sqrt{gd}$. If now a low obstacle is put in the stream, it will flow over it without changing the condition upstream. If the height of the obstacle is gradually increased, a hydraulic jump will finally appear, which will be stationary somewhere over the obstacle or somewhere upstream from it. The discharge over the obstacle as well as the condition far upstream will remain the same. As the height of the obstacle becomes near the conjugate depth of the supercritical stream, the hydraulic jump can no longer be stationary, because the height of water over the obstacle is insufficient for the fluid to have the same discharge as upstream. The hydraulic jump will then travel upstream and the influence of the obstacle will eventually be felt far upstream. But there still is no complete blocking. As the height of the obstacle becomes much more than the conjugate depth of the original supercritical stream, not only will the hydraulic jump travel upstream but there will be no flow past the obstacle. Blocking has occurred for a supercritical flow far upstream with a hydraulic jump moving upstream.

Imagine a reservoir from which the flow originates. The moving hydraulic jump will eventually "drown out" all the supercritical region and the entire flow from the reservoir downstream will be subcritical. Then the water level everywhere will rise, and, as it rises above the maximum height of the obstacle, there will again be a discharge downstream from the obstacle, if its maximum elevation is below that of the water level in the reservoir. If the obstacle becomes higher than the water level in the reservoir, there will be no flow anywhere, and final blocking has occurred.

If the stream is subcritical to start with, so that U is less than $\sqrt{gd}$, then any obstacle, however low, will affect the condition far upstream. But there will be no hydraulic jump for any low obstacle. As the obstacle becomes gradually higher and higher, the water level far upstream will rise, but the discharge will decrease. Finally, as the obstacle becomes as high as the total head of the original stream, there is no flow over the obstacle, and blocking has occurred, as mentioned in the last paragraph.

From this discussion it is evident that an obstacle can have an influence far upstream even in an originally supercritical stream, if it is high enough, that it always has an influence far upstream in a subcritical stream, and that in any case it can block the fluid but does not do so unless it is sufficiently high. If now the fluid is supposed to be stratified, similar conclusions can be expected, except that now it is possible for part of the fluid (near the bottom) to be blocked, while the top part still flows. Although here the discharge over the obstacle is controlled by gravity, whereas in the flow into the sink or over a barrier in a closed channel, as discussed in Sections 6 and 7 of this chapter, the discharge is prescribed and maintained by suction at the sink or downstream, there is much similarity between the two situations.

These ideas, which are very well known and have been restated by Yih [1969b], together with the classical result in hydraulics that the flow at the crest of a dam (or obstacle) is critical, are the basis of a calculation by Long [1972c] that illustrates, with the clarity we have come to expect of him, the upstream influence and partial blocking so well recognized by hydraulicians. Long considered a one-layer homogeneous fluid. The development for two layers, with a free surface and an interface, is considerably more complicated, and has been given by Houghton and Isaacson [1970], whose work on this very much discussed subject is the most relevant, important, and useful.

A rigorous and general treatment of the problem of blocking in a stratified fluid is not yet possible. But the foregoing discussion allows us to make the following qualitative assertions concerning the problem of blocking in a stratified fluid with some confidence.

(a) If the stream is supercritical with respect to all waves, internal or at the free surface, that is, if no infinitesimal waves can travel upstream, then low barriers will have no upstream influence, and therefore no blocking will occur. If the barrier is high, the original upstream condition may be affected, and partial blocking is possible.

(b) If the stream is subcritical with respect to internal waves, any barrier will have an upstream influence, but low barriers may not cause blocking. The change of upstream conditions when blocking is absent is obscured when we merely *assume* the final upstream conditions and seek a solution, as in Sections 3 through 7. Such upstream conditions can be considered to be the result of putting the barrier in an initial stream with somewhat different velocity and density distributions. Upstream influence is then not denied, merely by-passed. The same can be said of an originally supercritical flow when a high barrier is introduced, producing change in upstream conditions but not blocking. If the barrier is high in a subcritical or even originally supercritical stream, blocking will occur. One important conclusion to be drawn from this consideration is that solutions of (38) for $F < 1/\pi$ and for a low barrier are not spurious.

In a discussion of the blocking phenomenon, it is appropriate to mention the work of Drazin and Moore [1967]. They found that a solution is possible even for low values of the Froude number, however high the barrier (not necessarily the vertical wall). This is not surprising in view of the results shown in Figs. 13, 14, and 15. Their results showed, in the present notation, patterns of flow over a vertical wall for $F^{-1} = 1.5, 2.5$, and 3.5 and the height of the barrier equal to $d/4$ and $d/2$ (where d is the depth of channel). The cases for $d/2$ and especially the cases for $F^{-1} = 3.5$ and 4.5 and barrier height $d/4$ have very complicated flow patterns. These flows are not as likely to be realized as those obtained by Long [1953b] for low barriers even though Long's flow pattern also have eddies, which are verified by his experiments. They can be realized, of course, if the regions of closed streamlines are replaced by solid structures. But, as pointed out by Yih [1959d], separated flows with blocking are likely to result if this replacement is not done, as in Yih, O'Dell, and Debler [1962].

The question can only be settled convincingly by a study of the establishment of flow. Such a study would be extremely difficult. Before the work of Pao and Kao [1974], Trustrum [1964] used the Oseen approximation for such a study. Although the Oseen approximation is very much violated at the sink (which she considered) and far upstream as soon as blocking occurs, her conclusions are quite reasonable and in qualitative agreement with available theoretical or experimental evidences. Later Trustrum [1971] again used the Oseen model, this time to study two-dimensional stratified flows over an obstacle.

9. CLASS III: CHANNEL FLOWS AND LARGE-AMPLITUDE LEE WAVES WITH ANOTHER CLASS OF UPSTREAM CONDITIONS

Equation (15) will now be discussed. For definiteness, two-dimensional motion between two infinite horizontal plates spaced at distance d apart will again be considered. With the dimensionless variables defined by (18), (15) becomes, with $\alpha^2 = C_1 d^3$ and C_2 replaced by C,

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\Psi + \alpha^2\eta\Psi = Cd^2, \tag{74}$$

which is to be solved with the boundary conditions

$$\Psi = K_1 \quad \text{at} \quad \eta = 0, \qquad \Psi = K_2 \quad \text{at} \quad \eta = 1,$$

in which $\Psi = \psi'/U'd$, U' being some reference velocity.

The flow independent of x is determined by [see Long, 1958; Yih, 1960b; and the footnote after Eq. (34)]

$$\frac{d^2}{d\eta^2}\Psi_1 + \alpha^2\eta\Psi_1 = Cd^2, \tag{75}$$

in which Ψ_1 is the stream function for that flow. Equation (75) can be solved in a straightforward manner. Let

$$\Psi_1 = f(\eta)\phi(\eta) + K_1,$$

in which

$$f(\eta) = \eta^{1/2}J_{1/3}(\tfrac{2}{3}\alpha\eta^{3/2})$$

satisfies (75) with $C = 0$, and vanishes at $\eta = 0$, so that $\Psi_1 = K_1$ at $\eta = 0$. The equation for ϕ is then

$$f\phi'' + 2f'\phi' = Cd^2 - \alpha^2 K_1\eta, \tag{76}$$

in which primes on f and ϕ have been used to indicate differentiation with respect to η. After multiplying by f, (76) is immediately integrable:

$$\phi' = \frac{Cd^2\int_0^\eta f\,d\eta - \alpha^2 K_1\int_0^\eta \eta f\,d\eta}{f^2},$$

in which the constant of integration in the numerator is chosen to be zero to make ϕ' finite at $\eta = 0$. Another quadrature then produces ϕ. The final solution for Ψ_1 is

$$\Psi_1 = f(\eta)\left[\int_0^\eta \frac{Cd^2\int_0^\eta f\,d\eta - \alpha^2 K_1 \int_0^\eta \eta f\,d\eta}{f^2}\,d\eta + K'\right] + K_1, \quad (77)$$

in which K' is determined to satisfy the boundary condition

$$\Psi_1 = K_2 \quad \text{at} \quad \eta = 1.$$

The two boundary conditions for Ψ are satisfied by (77).

On the parallel flow can be superposed a flow governed by the equation

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\Psi_2 + \alpha^2 \eta \Psi_2 = 0, \quad (78)$$

the solution of which is, with D denoting an arbitrary constant associated with Ψ_2,

$$\begin{aligned}\Psi_2 = De^{a\xi}\cdot\beta(\alpha, b, \eta) &= De^{a\xi}\cdot(\eta - b)^{1/2} \\ &\times\{N_{1/3}[\tfrac{2}{3}\alpha(-b)^{3/2}]J_{1/3}[\tfrac{2}{3}\alpha(\eta - b)^{3/2}] \\ &- J_{1/3}[\tfrac{2}{3}\alpha(-b)^{3/2}]N_{1/3}[\tfrac{2}{3}\alpha(\eta - b)^{3/2}]\},\end{aligned} \quad (79)$$

in which $b = -a^2\alpha^{-7/3}$, a is a separation-of-variables constant, and J and N are Bessel and Neumann functions respectively. This solution satisfies the boundary condition $\Psi_2 = 0$ at $\eta = 0$. In order that the upper boundary be a streamline, $\beta(a, b, \eta)$ must satisfy the condition

$$\beta(\alpha, b, 1) = 0. \quad (80)$$

If C_1 is positive, α is real, and for positive α, (80) *may* have some real positive roots (or may not have any) for b (and hence imaginary roots for a), but not infinitely many such roots. However if C_1 is positive α can also be taken as negative. Then (80) has infinitely many real negative roots for b, corresponding to infinitely many real a's. Real a corresponds to local disturbances and imaginary a corresponds to waves in the fluid. The case of negative C_1 is more complicated, but can be treated similarly.

The flow represented by (79) has infinitely many stationary cells, because there are imaginary values of a. This flow can be superposed on that represented by Ψ_1. The resulting flow is a wavy flow toward the right

$$\Psi = \Psi_1 + \Psi_2.$$

If Cd^2/D is large, parallel flow dominates the waviness. If it is small, waviness is predominant. A solution of the type

$$\Psi = \Psi_1 + \sum_{n=1}^{\infty} A_n e^{a_n\xi}\beta(\alpha, b_n, \eta) \quad (81)$$

can be used to investigate channel flows or flows over a barrier, in the same way as indicated in Sections 7 and 8.

This concludes the discussion of the many exact solutions arising from Eq. (12).

10. COMPRESSIBLE FLUIDS WITH VARIABLE ENTROPY

In general, a compressible fluid with density stratification in the region of parallel flow is also stratified with respect to entropy. In the special case of homentropy, a flow started from rest will be irrotational, even if gravity is taken into account, as is well known. Because of compressibility the governing equation for the velocity potential for homentropic irrotational flows is much more complicated than the Laplace equation, and is nonlinear. If gravity is neglected, the well known Molenbroek-Tschapligin tranformation and the Legendre transformation provide powerful inverse methods for solving two-dimensional problems by linearizing the governing equation exactly. These transformations are not useful if gravity is taken into account, or if the flow is not two-dimensional, or else if the flow, though homentropic, is not irrotational. It would seem doubtful, then, that in the case of nonhomentropy there is much chance for exact solutions, particularly if gravity is taken into account. (If gravity effects are neglected, to each exact homentropic irrotational flow correspond infinitely many exact nonhomentropic flows.)

However, in the flow of air over mountain ranges, *dynamic* compressibility is not of great importance, because the prevailing velocities in the regions of interest are far below the local sound velocities. If dynamic compressibility is neglected, and if the entropy variation is slight, the equation governing steady two-dimensional flows is linear under certain circumstances, even if the amplitude of the motion is finite and the effect of gravity is taken into account. The solution of the linear equation is straightforward, and is useful for the study of lee waves behind mountain ranges.

The equation governing steady, two-dimensional nonhomentropic flows will now be presented. With u' and w' denoting u_1' and u_3' defined in (1.27), and with gravity force included, the equations of motion are

$$u'\frac{\partial u'}{\partial x} + w'\frac{\partial u'}{\partial z} = -\frac{1}{\rho'}\frac{\partial p'}{\partial x}, \tag{82}$$

$$u'\frac{\partial w'}{\partial x} + w'\frac{\partial w'}{\partial z} = -\frac{1}{\rho'}\frac{\partial p'}{\partial z} - g\lambda, \tag{83}$$

in which

$$\lambda = \text{constant} \cdot e^{-S/c_p}, \tag{84}$$

with S indicating entropy. The equation of continuity is now of the form

$$\frac{\partial(\rho' u')}{\partial x} + \frac{\partial(\rho' w')}{\partial z} = 0, \tag{85}$$

which permits the use of the stream function ψ, in terms of which

$$u' = \frac{1}{\rho'}\frac{\partial\psi'}{\partial z}, \qquad w' = -\frac{1}{\rho'}\frac{\partial\psi'}{\partial x}. \tag{86}$$

With

$$q'^2 = u'^2 + w'^2, \qquad \eta' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} = \frac{1}{\rho'}\nabla^2\psi' - \frac{1}{\rho'^2}\left(\frac{\partial\psi'}{\partial x}\frac{\partial\rho'}{\partial x} + \frac{\partial\psi'}{\partial z}\frac{\partial\rho'}{\partial z}\right), \tag{87}$$

(82) and (83) can be written

$$\frac{\eta'}{\rho'}\frac{\partial\psi'}{\partial x} = \frac{\partial}{\partial x}\left(\int\frac{dp'}{\rho'} + \frac{q'^2}{2}\right), \tag{88}$$

$$\frac{\eta'}{\rho'}\frac{\partial\psi'}{\partial z} = \frac{\partial}{\partial z}\left(\int\frac{dp'}{\rho'} + \frac{q'^2}{2}\right) + g\lambda. \tag{89}$$

Multiplying (88) by dx and (89) by dz and adding the results, we have

$$\frac{\eta'}{\rho'}\,d\psi' = dH - gz\,d\lambda, \tag{90}$$

in which

$$H = \frac{q'^2}{2} + gz\lambda + \int\frac{dp'}{\rho'} \tag{91}$$

is the Bernoulli quantity, and is a function of ψ' alone. Since λ depends only on the entropy, which is constant on a streamline, it is also a function of ψ' alone. Thus (90) can be written [Yih, 1960c; the corresponding equation in terms of ρ, η, and ψ was given by Dubreil-Jacotin, 1935 and Long, 1953c]

$$\frac{\eta'}{\rho'} + gz\frac{d\lambda}{d\psi'} = \frac{dH}{d\psi'} = h(\psi'), \tag{92}$$

or [Yih, 1960c]

$$\nabla^2\psi' - \frac{1}{\rho'}\left(\frac{\partial\psi'}{\partial x}\frac{\partial\rho'}{\partial x} + \frac{\partial\psi'}{\partial z}\frac{\partial\rho'}{\partial z}\right) + gz\rho'^2\frac{d\lambda}{d\psi'} = \rho'^2 h(\psi'). \tag{93}$$

The quantity ρ', defined in (1.27) can be expressed in terms of ψ', λ, H, and z by means of (91) and (87). Supplemented by this expression for ρ', (93) is the exact equation governing the nonhomentropic, nonhomenergetic* flows of an inviscid compressible fluid in a gravitational field.

11. HOMENTROPIC AND HOMENERGETIC FLOWS OVER GREAT HEIGHTS

Equation (93) is so complicated that it is very difficult to obtain a solution of it for arbitrary variations in entropy and specific energy. However, for

* A homenergetic flow is one with homogeneous specific energy or H.

homenergetic flows with homentropy, solutions can be easily constructed by inverse methods, even if these flows are over great heights.

If the specific energy and the entropy are constants, H and λ are constants. If the speed is everywhere not greater than one-tenth (say) of the local speed of sound, dynamic compressibility, represented by the term $q'^2/2$ in (91), can be neglected and a high degree of accuracy can still be maintained. Thus with ρ_0 and p_0 denoting the density and the pressure at a point $z = 0$ in the field of flow, $\lambda = 1$, and $\rho = \rho'$. With the term involving speed omitted, (91) becomes (since $p = p'$)

$$H = gz + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}, \tag{94}$$

which is merely the original Bernoulli equation, with the speed term omitted. Since

$$p = C\rho^\gamma, \qquad C = \frac{p_0}{\rho_0^\gamma}, \tag{95}$$

(94) can be solved for ρ:

$$\rho = \left[\frac{\gamma - 1}{C\gamma}(H - gz)\right]^{1/(\gamma-1)}. \tag{96}$$

Since now $\psi' = \psi$, (93) becomes

$$\nabla^2\psi + \frac{1}{\gamma - 1}\frac{g}{H - gz}\frac{\partial\psi}{\partial z} = 0, \tag{97}$$

which is due to Batchelor [1953]. The solution of (97) by the method of separation of variables is of the form

$$\psi = C(H - gz)^{\gamma/(\gamma-1)} + \sum_{n=1}^{\infty} A_n e^{\pm k_n x} f(k_n, z), \tag{98}$$

in which $f(k, z)$ satisfies the equation (with primes indicating ordinary differentiation)

$$f'' + \frac{1}{\gamma - 1}\frac{g}{H - gz}f' + k^2 f = 0, \tag{99}$$

and z varies from zero to H/g. Equation (99), together with the appropriate boundary conditions, determine the allowable values for k, which are denoted by k_n. The solution of (99) is

$$f(k, z) = \left(\frac{H}{g} - z\right)^m Z_m\left[k\left(\frac{H}{g} - z\right)\right], \tag{100}$$

in which

$$m = \frac{\gamma}{2(\gamma - 1)} \qquad (= 7/4 \text{ for diatomic gases}), \tag{101}$$

and

$$Z_m = J_m + B(k)\, N_m, \tag{102}$$

J_m and N_m being the Bessel function and the Neumann function of mth order, respectively, and $B(k)$ being a constant dependent only on k.

If there is a very stable (and therefore nonhomentropic) layer which starts at $z = d$, then the vertical motion across that elevation is very much inhibited, and the stream function can be assumed to be constant at $z = d$ for a first approximation. In that case the two conditions

$$Z_m\left(\frac{kH}{g}\right) = 0 \qquad \text{and} \qquad Z_m\left[k\left(\frac{H}{g} - d\right)\right] = 0 \tag{103}$$

determine the eigenvalues k_n and the associated constants $B(k_n)$. With the general solution given by (98), the inverse method of Yih presented in Section 8 can be applied. If, in addition, the functions

$$\cos\frac{n\pi x}{b}\cdot\left(\frac{H}{g} - z\right)^m\left[\operatorname{ber}_m\frac{n\pi}{b}\left(\frac{H}{g} - z\right) + C\operatorname{bei}_m\frac{n\pi}{b}\left(\frac{H}{g} - z\right)\right]$$

(in which C is a constant that makes the bracket vanish at $z = d$) are utilized, Long's inverse method can again be used. The quantity b is again half length of a symmetric barrier. (Since in this case there are no lee waves, the final barrier obtained from an assumed symmetric barrier will be symmetric.)

12. FLOWS WITH SLIGHT STRATIFICATION IN ENTROPY AND SPECIFIC ENERGY

If the variations in H and λ are small, and dynamic compressibility is neglected, (91) can be written, with λ replaced by 1,

$$\frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0}\left(\frac{p}{\rho}\right)^{(\gamma-1)/\gamma} = \int\frac{dp'}{\rho'} = H - gz, \tag{104}$$

or

$$\frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0^\gamma}(\rho')^{\gamma-1} = H - gz, \tag{105}$$

so that, with H treated as constant here,

$$\frac{1}{\rho'}\frac{\partial\rho'}{\partial x} = 0, \qquad \frac{1}{\rho'}\frac{\partial\rho'}{\partial z} = -\frac{1}{\gamma - 1}\frac{g}{H - gz}. \tag{106}$$

With

$$K = \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0^\gamma}, \tag{107}$$

(93) becomes [Yih, 1960c]

$$\nabla^2\psi' + \frac{1}{\gamma - 1}\frac{g}{H - gz}\frac{\partial\psi'}{\partial z} + gz\left(\frac{H - gz}{K}\right)^{2/(\gamma-1)}\frac{d\lambda}{d\psi'}$$
$$= \left(\frac{H - gz}{K}\right)^{2/(\gamma-1)} h(\psi'). \tag{108}$$

If λ and h are arbitrary functions of ψ', this equation is of course nonlinear. To search for solvable cases (linear cases), we may set

$$\frac{d\lambda}{d\psi'} = A\psi' + B, \qquad h(\psi') = C\psi' + D \tag{109}$$

as in Section 3, and proceed as before. Since ψ' can be changed by an arbitrary constant, there are six cases:

(a)	$A \neq 0,$	$B = 0,$	$C = 0,$	$D = 0,$
(b)	$A \neq 0,$	$B = 0,$	$C = 0,$	$D \neq 0,$
(c)	$A \neq 0,$	$B = 0,$	$C \neq 0,$	$D = 0,$
(d)	$A = 0,$	$B \neq 0,$	$C = 0,$	$D = 0,$
(e)	$A = 0,$	$B \neq 0,$	$C = 0,$	$D \neq 0,$
(f)	$A = 0,$	$B \neq 0,$	$C \neq 0,$	$D = 0.$

Changing z by a constant reduces case (c) to (a), and (e) to (d). Therefore there are only four essentially different cases. Although the solutions are less simple than those in Section 3 because the equation is less simple, the development is quite the same. Although in the calculation for ρ' some simplifying assumptions have been made, (108) provides the basis for a theory which is very useful for dealing with slight variations in λ and H, because large amplitudes (in height) of motion are explicity allowed, and, in spite of the approximation made in calculating ρ', (108) has *not* been obtained purely by perturbation.

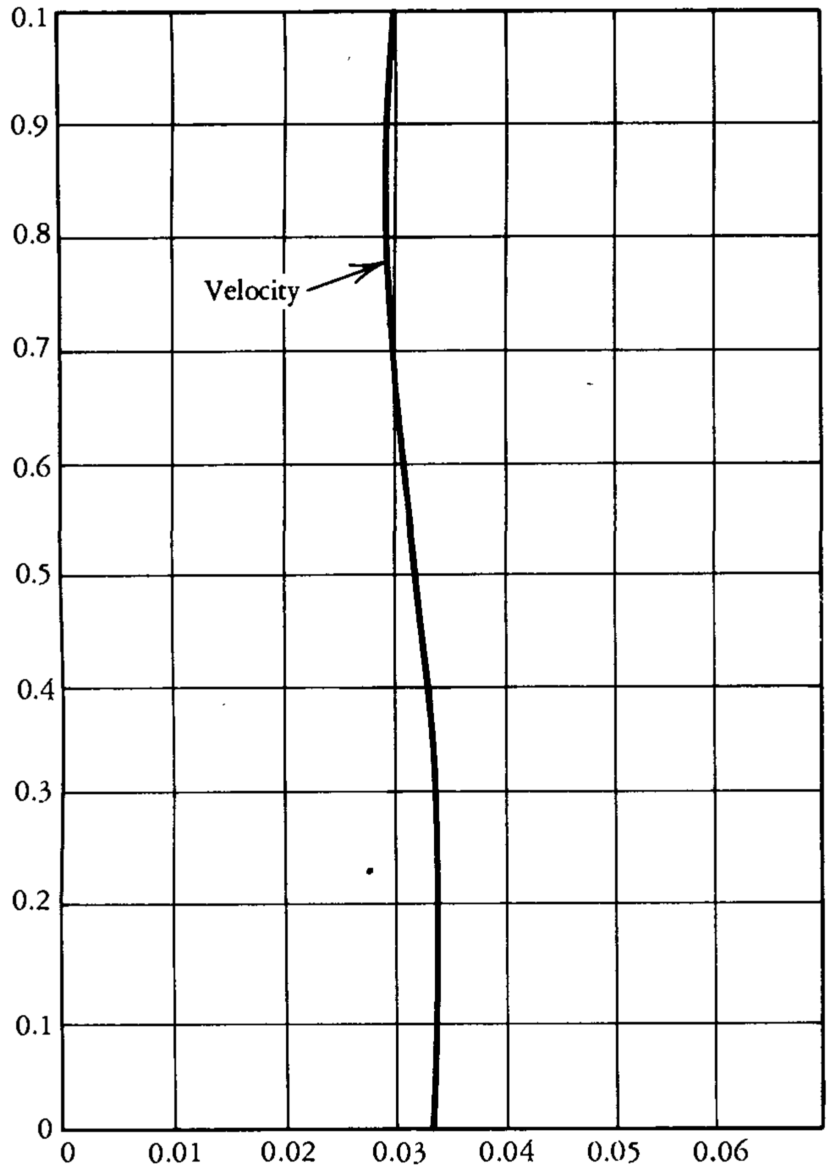
Alfons Claus [1961 and 1964] has investigated some of the linear cases of (108), by the methods of Long [1953b] and Yih [1960b], and has found the following results:

(1) The possible wind profiles and entropy stratifications for the linear cases have enough flexibility in them to approximate realistic atmospheric conditions rather closely.

(2) The number of lee-wave components increases as the wind speed decreases, and as the entropy gradient increases.

(3) The flow of a compressible fluid with stratification in entropy and the flow of an incompressible fluid with stratification in density are similar in general characteristics, provided the upstream distribution in potential density in the former is identical with the upstream density distribution in the latter, and the wind profiles are identical or nearly so. However, in detail the streamline spacings in the former are more varied than in the latter and the lee waves more pronounced. In particular, the wave amplitude along a streamline near the ground can be several times as large in the case of the compressible fluid as in the case of the incompressible fluid. See Figs. 23 and 24 for the wind profiles (practically identical) and density stratifications shown in Figs. 21 and 22.

As the air ascends it expands and is cooled. If it contains water vapor the cooling may cause it to condense into droplets. Thus cloud bars may form at



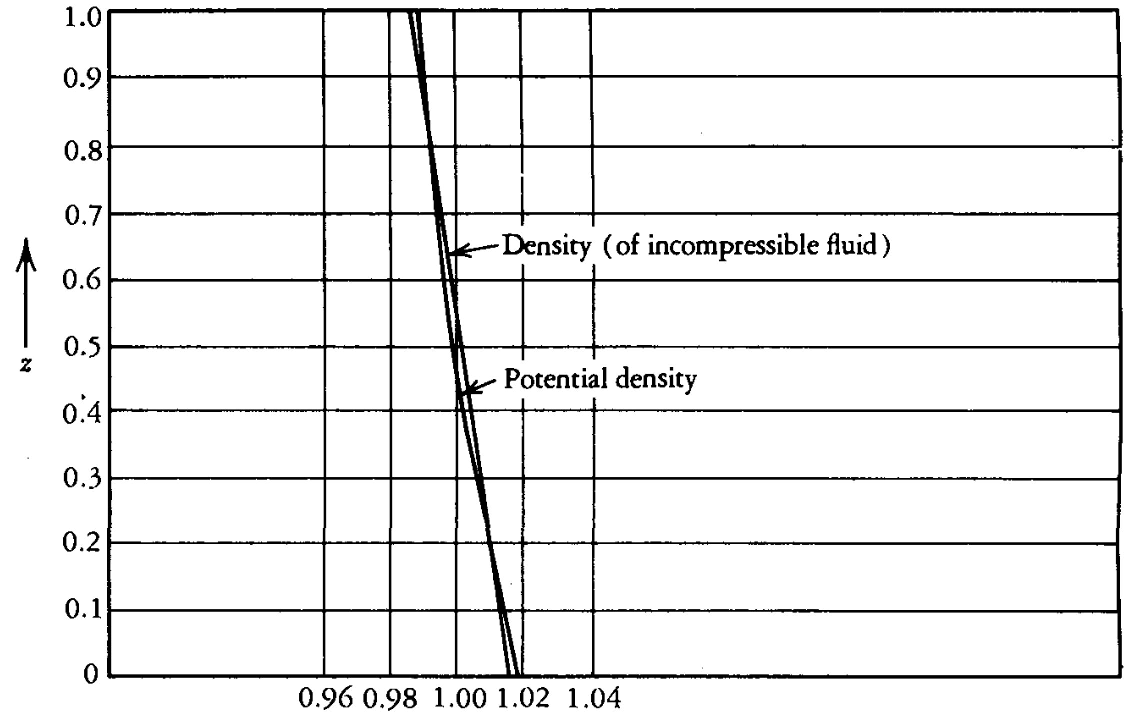


FIGURE 21. (Left) Upstream velocity (in terms of $\sqrt{gd}$) for incompressible fluid and for compressible fluid—almost identical for both cases. Figure 22. (Above) Approximation of the stratification in potential density in a compressible fluid by the density stratification in an incompressible fluid. All densities are expressed in terms of a reference density ρ_0. Both stratifications lead to linear governing equations (after Claus [1961 and 1964]). (*J. Fluid Mech.*, *19*, *part 2*. *Courtesy of the Cambridge Univ. Press.*)

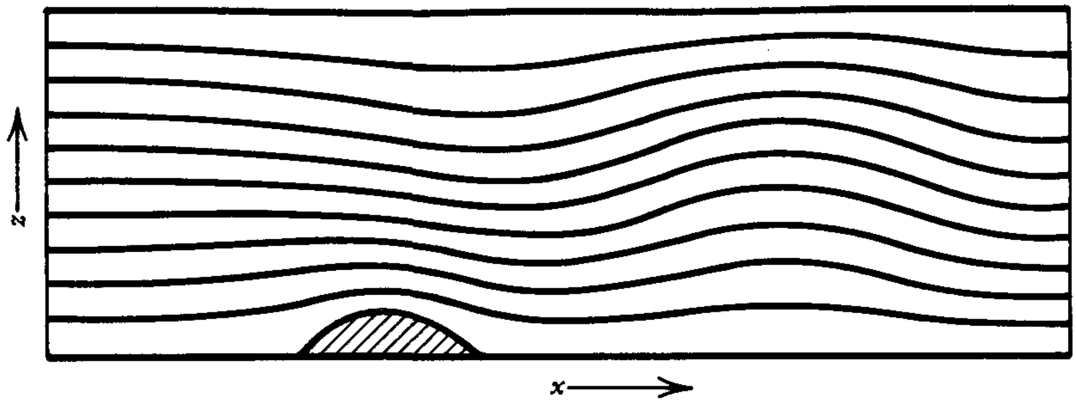


FIGURE 23. Comparison between the flow of an incompressible fluid with that of a compressible fluid. Incompressible case (after Claus, [1961 and 1964]). (*J. Fluid Mech., 19, part 2. Courtesy of the Cambridge Univ. Press.*)

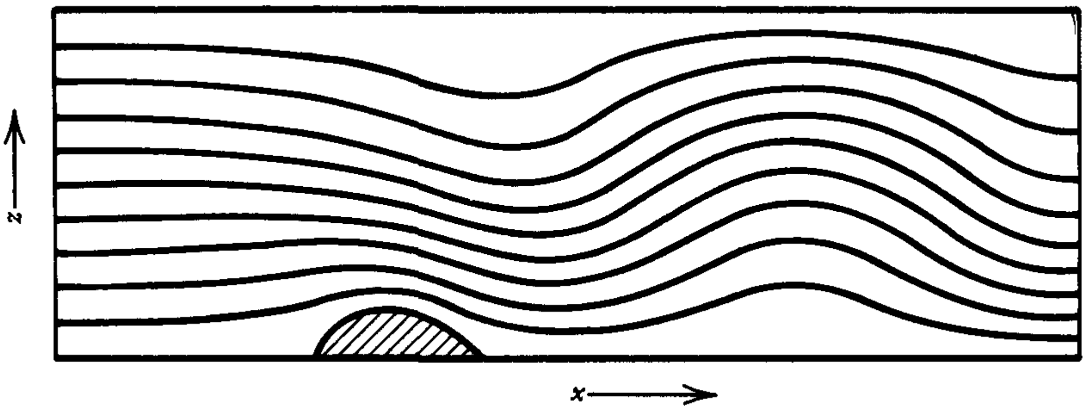


FIGURE 24. Comparison between the flow of an incompressible fluid with that of a compressible fluid. Compressible case (after Claus [1961 and 1964]). (*J. Fluid Mech., 19, part 2. Courtesy of the Cambridge Univ. Press.*)

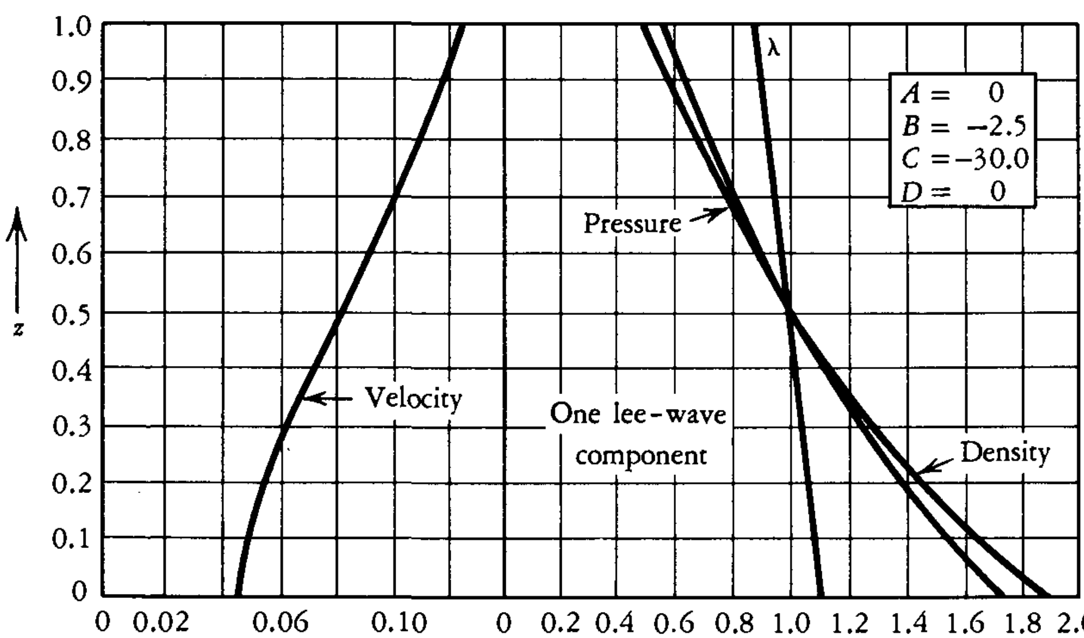


FIGURE 25. Upstream conditions leading to a flow pattern with one lee-wave component (after Claus [1961 and 1964]). The velocity is in terms of $\sqrt{gd}$, the pressure is in terms of a reference pressure p_0, and the density is in terms of a reference density ρ_0. (*J. Fluid Mech., 19, part 2. Courtesy of the Cambridge Univ. Press.*)

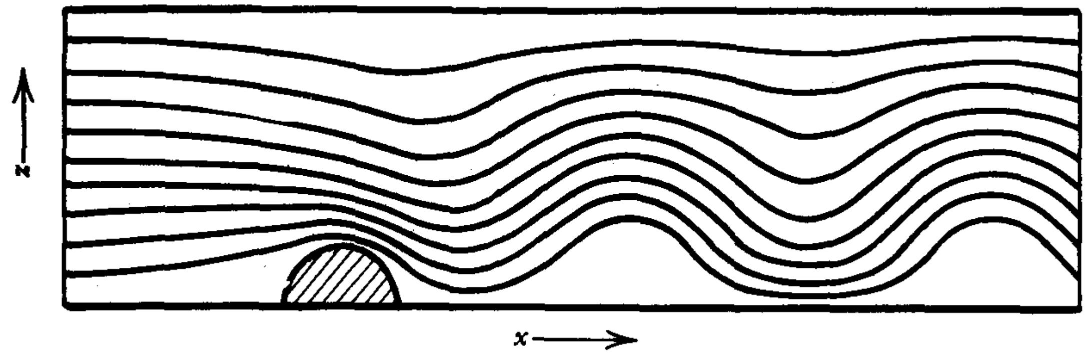


FIGURE 26. Flow pattern with one lee-wave component (after Claus [1961 and 1964]). (*J. Fluid Mech., 19, part 2. Courtesy of the Cambridge Univ. Press.*)

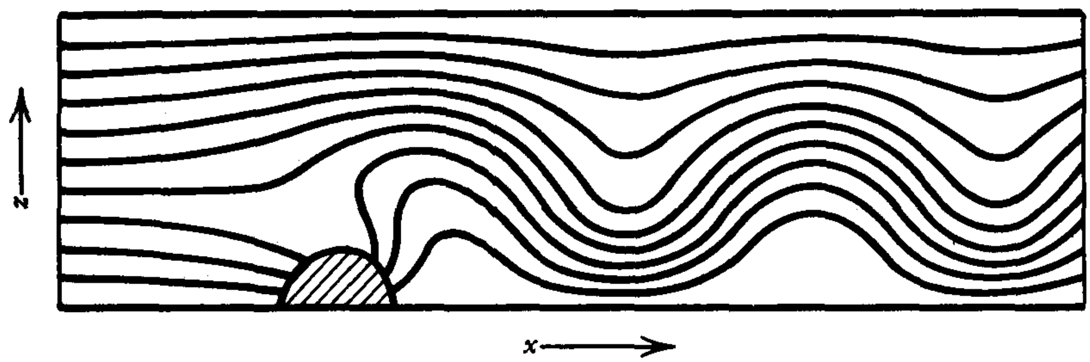


FIGURE 27. Isotherms in the flow with one lee-wave component (after Claus [1961 and 1964]). (*J. Fluid Mech., 19, part 2. Courtesy of the Cambridge Univ. Press.*)

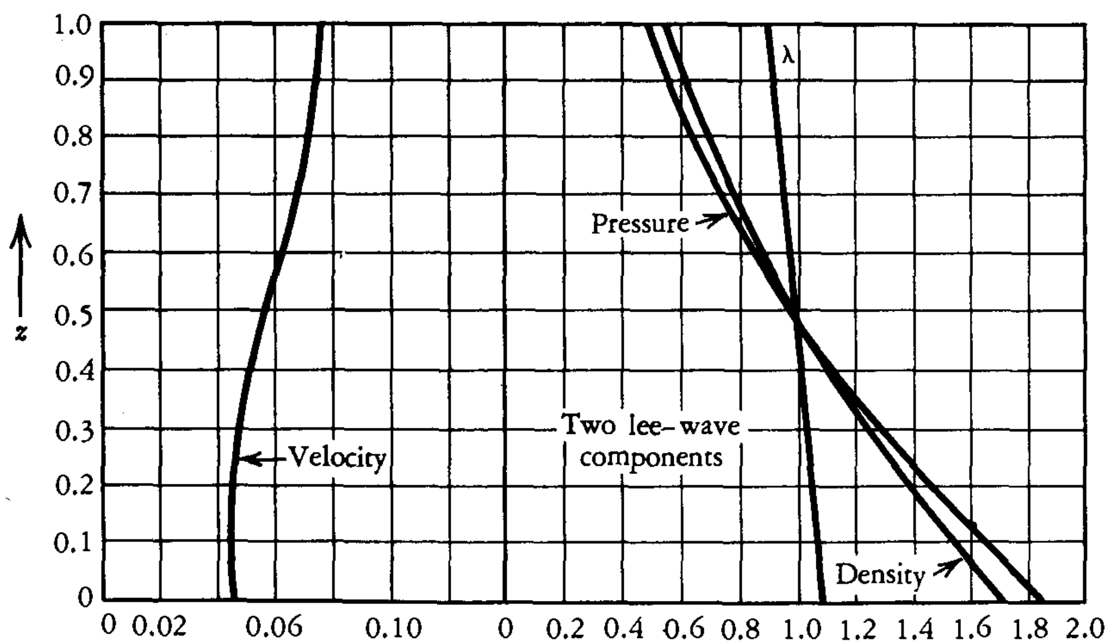


FIGURE 28. Upstream conditions leading to a flow pattern with two lee-wave components (after Claus [1961 and 1964]). The velocity is in terms of $\sqrt{gd}$, the pressure is in terms of a reference pressure p_0, and the density is in terms of a reference density ρ_0. (*J. Fluid Mech., 19, part 2. Courtesy of the Cambridge Univ. Press.*)

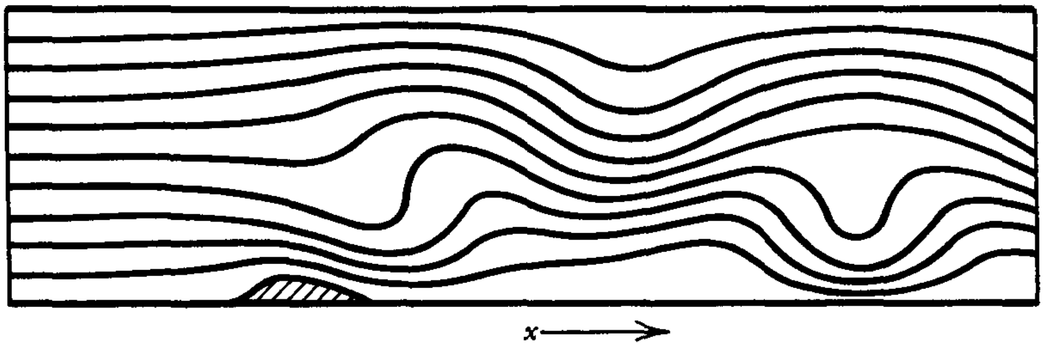


FIGURE 29. Flow pattern with two lee-wave components (after Claus [1961 and 1964]). (*J. Fluid Mech., 19, part 2. Courtesy of the Cambridge Univ. Press.*)

the crests of the lee waves, and be an indication of their existence. Claus has shown an isotherm pattern (Fig. 27) which corresponds to the flow pattern (Fig. 26) with one lee-wave component, with upstream conditions given in Fig. 25. The flow pattern for two lee-wave components is shown in Fig. 29, corresponding to upstream conditions given in Fig. 28.

13. AXISYMMETRIC FLOWS

We now turn to a brief discussion of axisymmetric flows of a nonhomogeneous fluid which may have a bearing on such engineering devices as centrifuges and such meteorological phenomena as tornadoes.

If u, v, and w denote velocity components in cylindrical coordinates (r, θ, z), and

$$\frac{D}{Dt} = u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}$$

for axisymmetric steady flows, the equations of motion are, for flows without swirl ($v = 0$),

$$\rho\frac{D}{Dt}(u, w) = -\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right)p - (0, g\rho).$$

For an incompressible fluid

$$\frac{D\rho}{Dt} = 0, \tag{110}$$

and the equations of motion can be written in the form

$$\rho_0\frac{D}{Dt}(u', w') = -\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right)p - (0, g\rho), \tag{111}$$

or

$$\rho_0 w'\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial r}\right) = -\frac{\partial}{\partial r}\left[p + \frac{\rho_0(u'^2 + w'^2)}{2}\right], \tag{112}$$

$$-\rho_0 u'\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial r}\right) = -\frac{\partial}{\partial z}\left[p + \frac{\rho_0(u'^2 + w'^2)}{2}\right] - g\rho, \tag{113}$$

in which again

$$(u', w') = \sqrt{\frac{\rho}{\rho_0}}(u, w).$$

As a consequence of (110),

$$\frac{\partial(ru')}{\partial r} + \frac{\partial(rw')}{\partial z} = 0,$$

so that

$$u' = -\frac{1}{r}\frac{\partial\psi'}{\partial z}, \qquad w' = \frac{1}{r}\frac{\partial\psi'}{\partial r}.$$

Equations (111) and (112) can be multiplied by dr and dz, respectively, and added to produce

$$\frac{\rho_0}{r}\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial r}\right) d\psi' = -d\left(p + \frac{\rho_0(u'^2 + w'^2)}{2}\right) - g\rho\, dz$$
$$= -dH + gz\, d\rho,$$

or

$$\frac{1}{r^2}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi' + \frac{g}{\rho_0}\frac{d\rho}{d\psi'}z = \frac{1}{\rho_0}\frac{dH}{d\psi'} = h(\psi'), \tag{114}$$

in which

$$H = p + \frac{\rho(u^2 + w^2)}{2} + g\rho z$$

is the Bernoulli quantity.

The corresponding equation for spherical coordinates (R, θ, ϕ) is

$$\frac{1}{R^2 \sin^2\theta}\left(\frac{\partial^2}{\partial R^2} - \frac{\cot\theta}{R^2}\frac{\partial}{\partial\theta} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\right)\psi' + \frac{g}{\rho_0}\frac{d\rho}{d\psi'}R = h(\psi'), \tag{115}$$

if gravity is constant but in the direction of decreasing R. Equations (114) and (115) are due to Yih [1957]. More generally, the Ω in (1.31) replaces gR.

Equation (114) is linear if $d\rho/d\Psi$ and $h(\Psi)$ are linear in Ψ. There are again three classes of solutions. Only the class corresponding to constant $d\rho/d\psi'$ and constant $h(\psi')$ will be discussed here. If U' is a reference velocity and d is a reference length, and

$$\Psi = \frac{\psi'}{U'd}, \qquad F^{-2} = -\frac{1}{\rho_0}\frac{d\rho}{d\psi'}\frac{gd^3}{U'}, \qquad \xi = \frac{r}{d}, \qquad \eta = \frac{z}{d}, \qquad h(\psi') = a^2 d^{-4},$$

one subclass of solution of (114) is

$$\Psi = \tfrac{1}{2}\xi^4(a^2 + F^{-2}\eta) + f_1(\eta) + C_1\xi^2(a^2 + F^{-2}\eta), \tag{116}$$

and another

$$\Psi = \frac{F^4\xi^2}{6}(a^2 + F^{-2}\eta)^3 + f_2(\xi)(a^2 + F^{-2}\eta) + f_3(\xi). \qquad (117)$$

If $f_1(\eta)$ is taken to be $C(a^2 + F^{-2}\eta)$ and $C_1 = 0$, (116) represents a stagnation-point flow, with the plane $\eta = -a^2F^2$ as a stream surface. If $f_1(\eta) = 0$ and C_1 is not zero, (116) represents a flow in and out of a circular bucket. By choosing $f_2(\xi)$ and $f_3(\xi)$ to be proportional to ξ^2, (117) can be made to represent models of gravitational convection with one, two, or three horizontal stream surfaces. But the condition at $\xi = \infty$ is not realistic.

If the flow is axially symmetric and has a swirl, the governing equation is a combination of (114) and (6.50), and is, in cylindrical coordinates,

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi' + \frac{g}{\rho_0}\frac{d\rho}{d\psi'}r^2 z + \frac{f'(\psi')}{2} = r^2 h(\psi'),$$

in which

$$f'(\psi') = \frac{d}{d\psi'}f(\psi'),$$

and

$$f(\psi') = (v'r)^2,$$

v' being the swirling component of the associated flow.

For steady, isentropic, axisymmetric flows of a swirling nonhomentropic fluid in a gravitational field, the governing equation is [Yih 1960c]

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi' - \frac{1}{\rho'}\left(\frac{\partial\rho'}{\partial r}\frac{\partial\psi'}{\partial r} + \frac{\partial\rho'}{\partial z}\frac{\partial\psi'}{\partial z}\right) + \frac{\rho'^2}{2}\frac{df}{d\psi'} + g\rho'^2 r^2 z\frac{d\lambda}{d\psi'}$$
$$= r^2\rho'^2\frac{dH}{d\psi'},$$

with $f(\psi') = (v'r)^2$, and with ψ', ρ', λ, and H as defined in Section 10.

14. PROGRESSIVE WAVES OF PERMANENT FORM IN CONTINUOUSLY STRATIFIED FLUIDS

Waves of permanent form progressing in a stratified fluid otherwise at rest can be studied most conveniently by adopting a frame of reference moving with the waves and thereby making the flow independent of time. The equation governing steady two-dimensional flows of a continuously stratified incompressible fluid is (10) or (11).

Whenever (10) or (11) is exactly linear and admits solutions representing waves, such solutions are for waves of any amplitude in a continuously stratified fluid. However, there is some artificiality associated with the up-stream conditions permitting (10) or (11) to be exactly linear: There is always a nonuniform velocity distribution of a parallel flow to which the solutions for waves are superposed, and this nonuniformity is always there no matter how small the amplitude of the waves. In other words, a current of nonuniform velocity must always be present for (10) or (11) to be exactly fluid, it is well known that the solution for the flow in the frame of reference moving with the waves must consist of a flow with uniform velocity and a flow representing the waves, provided the amplitude of the waves is very small. Consequently, the exactly linear cases cannot correspond to progressive infinitesimal waves, and by extension cannot correspond to waves of finite amplitude (which are adjacent to infinitesimal waves) progressing in an otherwise quiescent fluid.

Yih [1974c] investigated waves of permanent form progressing in an otherwise quiescent fluid. Since the differential equation (10) or (11) is bound to be nonlinear for such waves, there is no advantage in using (10). The problem then is first to determine the functions $d\rho/d\psi$ and $h(\psi)$ in (11). This is now not as simple a task as in the study of lee waves, since the waves extend from minus infinity to infinity in the x-direction, and there is no parallel flow far upstream. We shall use a series expansion in powers of the amplitude and show how the differential equation can be determined at each stage of the approximation, as well as the eigenfunctions and eigenvalues (for the wave velocity). In arriving at the mathematical solution of the problem, we are also able to assess the degree of validity of the Boussinesq approximation, which has been used so very extensively. Our solution is for any wave number, any stratification, and any mode of the internal wave for a given wave number. (The precise meaning of the density stratification will be discussed later in this section. The need for such a discussion arises from the lack of a section far upstream where the flow is parallel.) Special attention will be given to the important and realistic cases of weak density gradients.

Before formulating and solving our problem we note that this problem has already been studied by Thorpe [1968a] in an extensive paper. However, in Thorpe's work the Boussinesq approximation has been used and the effect of amplitude on the wave velocity has not been determined, and indeed there is no indication how that effect can be determined. Furthermore, in all previous work on waves in a continuously stratified fluid, Thorpe's included, it has never been pointed out that it is necessary to determine or specify the density distribution in the fluid when it is allowed to quiet down, so that we know for what fluid the problem is solved. This will be an important aspect of our solution to be presented in this section.

We note that the method of determining the eigenvalue of the wave velocity at each stage of the approximation involves the requirement that the inhomogeneous part $g(y, \Delta\lambda)$ of the equation

$$L_\lambda f(z) = g(y, \Delta\lambda)$$

be orthogonal to the eigenfunction $f_0(y)$ satisfying

$$L_\lambda f_0(z) = 0$$

and the boundary conditions, L_λ being a linear operator containing the eigenvalue λ, and $\Delta\lambda$ being the correction to λ that is necessary at any particular stage of the approximation under consideration. This technique is a powerful tool in the study of linear eigenvalue problems.

The problem is to study the flow due to waves progressing in a continuously stratified fluid otherwise at rest. The fluid is specified by its density distribution when waves are absent, and this same fluid is always under consideration when waves are present.

The first task is to specify the general form of (11) that applies to our problem, whatever the density distribution of the fluid. To this end we recognize that that form must satisfy the following requirements: When the amplitude of the waves is very small, it must reduce to the well known linear equation (for the moving frame of reference) governing infinitesimal waves, and as a consequence it must allow a parallel flow of uniform velocity (equal to the wave velocity) when the amplitude of the waves is reduced to zero. Keeping this velocity-distribution requirement in mind, we see that the form sought is

$$\psi_{xx} + \psi_{zz} + \frac{1}{\rho}\frac{d\rho}{d\psi}\left(\frac{\psi_x^2 + \psi_z^2}{2} + gz\right) = \frac{1}{\rho}\frac{d\rho}{d\psi}\left(\frac{g}{c}\psi + \frac{c^2}{2}\right), \tag{118}$$

where c is the wave velocity. Indeed, this equation has already been given by Davis and Acrivos [1967b]. A flow of uniform velocity c in the direction of increasing x has been superposed on waves progressing in the opposite direction to make the flow steady. Note that

$$\psi = cz \tag{119}$$

is always a possible solution, with no waves present at all, and that when the amplitude is small (118) indeed reduces to the wave equation for infinitesimal waves, upon replacing ψ in $d\rho/d\psi$ by cz. Note, however, that $d\rho/d\psi$ for finite-amplitude waves is not known and cannot be specified *a priori*. Indeed, it has to be determined and redetermined at succeeding stages of approximation, with the density-distribution requirement in mind.

We shall consider two-dimensional internal waves bounded by the horizontal boundaries $z = 0$ and $z = d$, so that

$$\psi = \text{constant} \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d. \tag{120}$$

Equations (118) and (120) constitute the eigenvalue problem, with c as the eigenvalue and ψ as the eigenfunction.

It will be convenient to use the following dimensionless variables:

$$\Psi = \psi/cd, \qquad \xi = x/d, \qquad \eta = z/d, \qquad F^2 = c^2/gd, \qquad \hat{\rho} = \rho/\rho_0, \tag{121}$$

where ρ_0 is the density at $\eta = 0$, and F is the Froude number. Note that the F defined in (121) is not the F used in this chapter up to now, which is denoted by F_i below. In terms of these dimensionless variables, (121) becomes, after the caret on $\hat{\rho}$ has been dropped,

$$\Psi_{\xi\xi} + \Psi_{\eta\eta} + \frac{1}{\rho}\frac{d\rho}{d\Psi}\left(\frac{\Psi_\xi^2 + \Psi_\eta^2}{2} + F^{-2}\eta\right) = \frac{1}{\rho}\frac{d\rho}{d\Psi}\left(F^{-2}\Psi + \frac{1}{2}\right). \quad (122)$$

The boundary conditions becomes

$$\Psi = \text{constant} \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \eta = 1. \quad (123)$$

In what follows we shall consider a fluid which, when at rest, has the dimensionless-density distribution

$$\rho = \exp(-\beta\bar{\eta}) \quad (124)$$

where $\bar{\eta}$ is η of the constant-density surfaces when no waves are present. The theory developed for the particular distribution applies to any general density distribution when the necessary changes are made to account for the density distribution specified. The "densimetric" or "internal" Froude number F_i is defined by

$$\beta F_i^2 = F^2, \quad (125)$$

and is the F used in this chapter before this section.

We shall expand Ψ and F_i^2 in the series (A = amplitude)

$$\Psi = \Psi_0 + A\Psi_1 + A^2\Psi_2 + \cdots, \quad (126)$$

$$F_i^{-2} = G_0 + AG_1 + A^2G_2 + \cdots. \quad (127)$$

It is obvious that

$$\Psi_0 = \eta, \quad (128)$$

which satisfies (122) and (124) exactly. At this stage of approximation η is $\bar{\eta}$. Hence, from (124) and (128) we have

$$\frac{1}{\rho}\frac{d\rho}{d\Psi} = \frac{1}{\rho}\frac{d\rho}{d\Psi_0} = -\beta, \quad (129)$$

and (122) becomes, upon collecting terms of the power A,

$$(\Psi_1)_{\xi\xi} + (\Psi_1)_{\eta\eta} - \beta(\Psi_1)_\eta = -G_0\Psi_1, \quad (130)$$

which is the equation governing infinitesimal waves. The solution of (130) with the boundary conditions (123), in which Ψ_1 is used for Ψ, gives

$$\Psi_1 = \exp(\beta\eta/2)\sin n\pi\eta \sin k\xi \quad (131)$$

and

$$G_0 = n^2\pi^2 + k^2 + \beta^2/4, \quad (132)$$

where k is the wave number and n is an integer indicating the mode. For any given k there are infinitely many modes corresponding to positive integral values of n. The larger n is, the "higher" the mode.

At this stage it is necessary to reevaluate $d\rho/d\Psi$, to see whether a new evaluation should be used for the next approximation. For this purpose we take two terms on the right-hand side of (126), and rewrite it as

$$\eta = \Psi - A \exp(\beta\eta/2) \sin n\pi\eta \sin k\xi. \tag{133}$$

Upon successive iterations we obtain, from (133), the result

$$\begin{aligned}\eta = \Psi &- A \exp(\beta\Psi/2) \sin n\pi\Psi \sin k\xi \\ &+ \tfrac{1}{4}A^2 \exp \beta\Psi\,(n\pi \sin 2n\pi\Psi + \beta \sin^2 n\pi\Psi)(1 - \cos 2k\xi) + O(A^3),\end{aligned} \tag{134}$$

which, upon averaging with respect to ξ, gives

$$\bar{\eta} = \Psi + \tfrac{1}{8}A^2 \exp \beta\Psi\,(\beta - \beta \cos 2n\pi\Psi + 2n\pi \sin 2n\pi\Psi) + O(A^4), \tag{135}$$

because the average value of the terms of $O(A^3)$ is zero. Note that the averaging is not an inexact process, for the definition of $\bar{\eta}$ demands exactly such an averaging process.

If we write

$$\frac{1}{\rho}\frac{d\rho}{d\Psi} = \frac{1}{\rho}\frac{d\rho}{d\bar{\eta}}\frac{d\bar{\eta}}{d\Psi} = -\beta\frac{d\bar{\eta}}{d\Psi},$$

and substitute this into (122), we have

$$\Psi_{\xi\xi} + \Psi_{\eta\eta} - \beta\frac{d\bar{\eta}}{d\Psi}\left(\frac{\Psi_\xi^2 + \Psi_\eta^2}{2} + F^{-2}\eta\right) = -\beta\frac{d\bar{\eta}}{d\Psi}\left(F^{-2}\Psi + \frac{1}{2}\right). \tag{136}$$

Using (135) in (136), we find that the terms of $O(A^2)$ in (135) introduce only terms of $O(A^3)$ in (136). Therefore, while we are determining the terms of $O(A^2)$ in Ψ, we need not yet make the correction for $d\rho/d\Psi$.

Keeping

$$\frac{1}{\rho}\frac{d\rho}{d\Psi} = -\beta,$$

then, we can proceed with the determination of Ψ_2. Using (126) to (128) and gathering terms of $O(A^2)$ in (122), we obtain

$$\nabla^2\Psi_2 - \beta(\Psi_2)_\eta + G_0\Psi_2 = \beta S - G_1\Psi_1, \tag{137}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}$$

and, after some reductions,

$$S = \exp \beta\eta\,(B_1 + B_2 \cos 2n\pi\eta + B_3 \sin 2n\pi\eta) - \exp \beta\eta\,(B'_1 + B'_2 \cos 2n\pi\eta + B_3 \sin 2n\pi\eta) \cos 2k\xi. \tag{138}$$

where

$$8B_1 = G_0, \qquad 32B_2 = 4n^2\pi^2 - 4k^2 - \beta^2, \qquad 8B_3 = n\pi\beta,$$
$$8B'_1 = n^2\pi^2 - k^2 + \beta^2/4, \qquad 32B'_2 = 4n^2\pi^2 + 4k^2 - \beta^2.$$

Solving (137) with the boundary conditions (123), we find that

$$G_1 = 0, \tag{139}$$

and

$$\Psi_2 = \beta\{B_1 f_1(\eta) + B_2 f_2(\eta) + B_3 f_3(\eta) - [B'_1 f_4(\eta) + B'_2 f_5(\eta) + B_3 f_6(\eta)] \cos 2k\xi\}, \tag{140}$$

where

$$\begin{aligned}
f_1(\eta) &= F_0^{-1} \exp \beta\eta,\\
f_2(\eta) &= (1/M) \exp \beta\eta\,[(G_0 - 4n^2\pi^2) \cos 2n\pi\eta + 2n\pi\beta \sin 2n\pi\eta],\\
f_3(\eta) &= (1/M) \exp \beta\eta\,[(G_0 - 4n^2\pi^2) \sin 2n\pi\eta - 2n\pi\beta \cos 2n\pi\eta],\\
f_4(\eta) &= (G_0 - 4k^2)^{-1} \exp (\beta\eta/2)\,[\exp (\beta\eta/2) - \cos \gamma\eta - a \sin \gamma\eta],\\
f_5(\eta) &= (1/N) \exp (\beta\eta/2)\,\{-P[\exp (\beta\eta/2) \cos 2n\pi\eta - \cos \gamma\eta - a \sin \gamma\eta]\\
&\quad + 2n\pi\beta \exp (\beta\eta/2) \sin 2n\pi\eta\},\\
f_6(\eta) &= (1/N) \exp (\beta\eta/2)\,\{-P \exp (\beta\eta/2) \sin 2n\pi\eta\\
&\quad - 2n\pi\beta[\exp (\beta\eta/2) \cos 2n\pi\eta - \cos \gamma\eta - a \sin \gamma\eta]\},
\end{aligned} \tag{141}$$

with

$$M = (G_0 - 4n^2\pi^2)^2 + 4n^2\pi^2\beta^2, \qquad N = P^2 + 4n^2\pi^2\beta^2, \qquad P = 4n^2\pi^2 + 4k^2 - G_0, \tag{142}$$

and

$$\gamma = \left(G_0 - 4k^2 - \frac{\beta^2}{4}\right)^{1/2}, \qquad a = (\exp (\beta/2) - \cos \gamma)(\sin \gamma)^{-1}. \tag{143}$$

The result (139) is obtained because the term βS does not contain the factor sin $k\xi$ or cos $k\xi$, and if G_1 did not vanish there would be no solution for Ψ_2. Indeed, since Ψ_1 contains sin $k\xi$, the particular solution of (137) to account for the term $-G_1\Psi_1$ must also contain it. Then, denoting that particular solution by $p_1(\eta)\sin k\xi$, we have $p_1(0) = 0 = p_1(1)$ and

$$Lp_1 = -G_1 \exp(\beta\eta/2) \sin n\pi\eta, \tag{144}$$

where

$$L = \frac{d^2}{d\eta^2} - \beta\frac{d}{d\eta} + G_0 - k^2. \tag{145}$$

On the other hand, (130) can be written as

$$L \exp(\beta\eta/2) \sin n\pi\eta = 0. \tag{146}$$

Multiplying (144) by $\exp(-\beta\eta/2)\sin n\pi\eta$ and (146) by $\exp(-\beta\eta)p_1$, integrating between zero and 1, using the boundary conditions for p_1, and taking the difference of the two integrated equations we have (139).

The functions f_4, f_5, and f_6 need a discussion because the γ defined in (143) may be zero or equal to $m\pi$, with m equal to an integer and less than n. In either case $\sin\gamma$ vanishes and the number a in (143) is infinite. The case $\gamma = 0$ is not really troublesome because the complimentary solutions $\exp(\beta\eta/2)\cos\gamma\eta$ and $\exp(\beta\eta/2)\sin\gamma\eta$ in f_4, f_5, and f_6 can be replaced by $\exp(\beta\eta/2)$ and $\eta\exp(\beta\eta/2)$, respectively, and with the constant a replaced by $\exp(\beta/2) - 1$. The case $\gamma = m\pi$, with m a nonzero integer less than n is much more significant. In this case

$$G_0 = n^2\pi^2 + k^2 + \frac{\beta^2}{4} = m^2\pi^2 + 4k^2 + \frac{\beta^2}{4} \tag{147}$$

and a look at (130) shows that whenever (147) holds the nth mode of internal waves with wave number k has the same wave velocity as the mth mode with wave number $2k$. In such a case it is necessary (in order to have internal waves of permanent form) to have first-order solutions in addition to Ψ_1 given by (131). These are solutions of (130) but with wave numbers 0 and $2k$, respectively, and are

$$\chi_1 = A\exp(\beta\eta/2)(a_1\sin\mu\eta + b_1\cos\mu\eta), \qquad \mu = \left(G_0 - \frac{\beta^2}{4}\right)^{1/2}, \tag{148}$$

$$\chi_2 = Aa_2\exp(\beta\eta/2)\sin m\pi\eta\cos 2k\xi, \tag{149}$$

in which a_2 is arbitrary. This means we can start with any two wave amplitudes for the wave trains of wave numbers k and $2k$. When (148) and (149) are included in Ψ_1 in (126) and (126) is substituted into (136), terms of $O(A^2)$ are collected, and then terms with factors $\sin k\xi$ and $\cos 2k\xi$ are separated, we obtain two equations:

$$L\theta_1 = a_1 g_1(\eta) + b_1 h_1(\eta) + a_2 g_2(\eta), \tag{150}$$

where L is the operator defined by (145), and

$$\left(\frac{d^2}{d\eta^2} - \beta\frac{d}{d\eta} + G_0 - 4k^2\right)\theta_2 = g_0(\eta) + a_1 a_2 g_3(\eta) + b_1 a_2 g_4(\eta). \quad (151)$$

By making the right-hand sides of (150) and (151) orthogonal to $\exp(\beta\eta/2)\sin n\pi\eta$ and $\exp(\beta\eta/2)\sin m\pi\eta$, respectively, we can determine a_1 and b_1. The solution θ_2 then replaces the coefficient of $\cos 2k\xi$ in (140) and the solution $\theta_1 \sin k\xi$ is the additional part which is now necessary.

Note that to f_1, f_2, and f_3 in (141) could be added solutions of the form (148). This is perhaps the reason that Thorpe [1968a] decided that parallel flows (currents), which can be added to the flow sought by him, can be determined only if one knows how the waves have been created. Actually, there does not appear to be any hope that these currents can really be determined from the circumstances of generation of the waves. Rather, the purpose is to isolate waves of fundamental wave number k as much as possible, avoiding adding any terms not having this wave number at all stages of approximation unless addition of such terms is necessary as particular solutions at higher approximations than the first, or even at the first approximation, as in the case $\gamma = m\pi$ discussed in the preceding paragraph. With this decision, the parallel currents are no longer arbitrary but uniquely determined, except in the case $\gamma = m\pi$ discussed above, where a_2 is arbitrary and hence so are a_1 and b_1. That exceptional case arises because waves of wave numbers k and $2k$ are no longer isolatable at the order $O(A^2)$.

One more point must be clarified before we go on to deal with terms of $O(A^3)$. If γ is not equal to $m\pi$ exactly but very near to it, the value of a in (143) can be very large, particularly if m is even. Then, Ψ_2 can be very large except for certain values of η, indeed very much larger than Ψ_1, and this seems strange and unreasonable. The difficulty is less serious when we remember that in the next stage of approximation large contributions are fed back in the same way to the term containing the factor $\sin k\xi$, and the amplitude will not be A but redetermined. Then, the ratio of Ψ_2 to Ψ_1 will not be so large. Difficulties of the kind discussed in this and the preceding paragraphs are typical of nonlinear problems. Their resolutions are never immediately clear and often, as in this case, require much additional work. Note that if $n = 1$ (for the first mode), then nonzero value of m does not exist, and the apparent difficulties discussed in the preceding paragraphs do not exist. For $n = 1$, however, there is still the possibility that the M defined in (142), which never vanishes, may nevertheless be of $O(\beta^2)$, thus making f_2 and f_3 in (141) very large. For the convenience of later discussions, we shall only deal with those values of k^2 which satisfy

$$3\pi^2 - k^2 = O(1), \quad (152)$$

for $n = 1$. For higher values of n, to avoid the case in which γ is equal or near

$m\pi$, we shall assume that (147) is never exactly or nearly satisfied, or that, with $n = 2$ and $m = 1$ to obtain the safe bound for k^2,

$$\pi^2 - k^2 = O(1). \tag{153}$$

If (153) is satisfied, then M defined in (142) is not small. Hence for $n > 1$, we impose (153) only. Since we have carried out our calculations only to $O(A^2)$ in (126) and (127), our conclusions will necessarily be valid only to this order. This imposes a limit (undetermined) on the magnitude of $|A|$.

Before going on with our calculation, it is now necessary to recalculate $\rho^{-1}\, d\rho/d\Psi$. Using (126) and (140), we obtain

$$\begin{aligned}\bar{\eta} = \Psi &+ \tfrac{1}{8}A^2 \exp(\beta\Psi)(\beta - \beta\cos 2n\pi\Psi + 2n\pi\Psi \sin 2n\pi\Psi) \\ &+ \beta A^2[B_1 f_1(\Psi) + B_2 f_2(\Psi) + B_3 f_3(\Psi)] + O(A^4).\end{aligned} \tag{154}$$

Substituting (154) into (136), using (126) and (127), and collecting terms of $O(A^3)$, we have

$$\left(\nabla^2 - \beta\frac{\partial}{\partial\eta} + G_0\right)\Psi_3 = T_1 + T_2 + T_3 - G_2\Psi_1, \tag{155}$$

where

$$\begin{aligned} T_1 &= \beta^2\phi'[B_1 f'_1 + B_2 f'_2 + B_3 f'_3 \\ &\quad + 0.5(B'_1 f'_4 + B'_2 f'_5 + B_3 f'_6)(\sin k\xi - \sin 3k\xi)] \\ &\quad + k^2\beta^2\phi(B'_1 f_4 + B'_2 f_5 + B_3 f_6)(\sin 3k\xi - \sin k\xi), \\ 8T_2 &= (\beta\phi' - G_0\phi)[\exp\beta\eta\,(\beta - \beta\cos 2n\pi\eta \\ &\quad + 2n\pi\sin 2n\pi\eta)]'\sin k\xi, \\ T_3 &= \beta(\beta\phi' - G_0\phi)(B_1 f'_1 + B_2 f'_2 + B_3 f'_3)\sin k\xi \end{aligned} \tag{156}$$

with

$$\phi = \exp(\beta\eta/2)\sin n\pi\eta. \tag{157}$$

The primes in (156) indicate differentiation with respect to η, and the functions f_1 to f_6 are functions of η.

To determine G_2, we again demand that the right-hand side of (155) be orthogonal to Ψ_1, that is to say, the sum J of the coefficients of $\sin k\xi$ of the right-hand side of (155) be orthogonal to ϕ in the sense that

$$\int_0^1 \exp(-\beta\eta)\, J\phi\, d\eta = 0. \tag{158}$$

After we determine G_2, we can find Ψ_3 by solving (155). We shall determine G_2 to see how the amplitude A affects the wave velocity, but we shall not attempt to determine Ψ_3. That determination is straightforward, but tedious.

Note that T_1 arises from the term containing $|\nabla\Psi|^2$ in (136), while T_2 and T_3 arise from the displacement of isopycnic lines as the result of Ψ_1 and Ψ_2, respectively. For all cases important in practice, β is very small; therefore, we shall consider β to be small. This will simplify the presentation of the results, although G_2 can be determined from (158) completely, with all terms included, in a straightforward although lengthy way. For small β, we shall show that the dominant term in G_2 comes from T_2.

If (152) is satisfied for $n = 1$ and (153) satisfied for $n > 1$, it can be shown that all the contributions to G_2 from T_1 (or rather those terms of T_1 that contain $\sin k\xi$) are $O(\beta^2)$. The same is true of the contributions from T_3. The details supporting these statements can be obtained in a straightforward manner but are very lengthy. We shall omit their presentation to save space.

Carrying out in full the contributions of T_2, we find that

$$\begin{aligned}8\int_0^1 \exp(-\beta\eta)\, T_2\phi\, d\eta = {} & 0.5\beta^4 I(\beta, n\pi) + \beta^2(n^2\pi^2 - 0.25\beta^2)[I(\beta, 3n\pi) \\ & - I(\beta, n\pi)] + n\pi\beta^3[R(\beta, n\pi) - R(\beta, 3n\pi)] \\ & + 0.5n\pi\beta\{\beta^2 I(\beta, 2n\pi) + (2n^2\pi^2 - 0.5\beta^2)I(\beta, 4n\pi) \\ & + 2n\pi\beta[R(\beta, 0) - R(\beta, 4n\pi)]\} \\ & - \tfrac{1}{2}G_0\{\beta^2 R(\beta, 0) + (4n^2\pi^2 - 2\beta^2)R(\beta, 2n\pi) \\ & + 4n\pi\beta I(\beta, 2n\pi) - 2n\pi\beta I(\beta, 4n\pi) \\ & + (0.5\beta^2 - 2n^2\pi^2)[R(\beta, 0) + R(\beta, 4n\pi)]\},\end{aligned} \tag{159}$$

in which the functions R and I are defined by

$$\begin{aligned}R(p, q) + iI(p, q) &= \int_0^1 \exp[(p + iq)\eta]\, d\eta \\ &= \frac{1}{p^2 + q^2}[p(e^p \cos q - 1) + q\, e^p \sin q] \\ &\quad + \frac{i}{p^2 + q^2}[p\, e^p \sin q - q(e^p \cos q - 1)].\end{aligned} \tag{160}$$

If $q = r\pi$ (r an integer),

$$R(p, r\pi) = \frac{p}{p^2 + r^2\pi^2}[e^p(-1)^r - 1],$$

$$I(p, r\pi) = \frac{r\pi}{p^2 + r^2\pi^2}[e^p(-1)^r - 1].$$

Thus

$$R(\beta, 0) = 1 + 0.5\beta + O(\beta^2),$$

and, with s equal to any positive integer,

$$R(\beta, (2s-1)\pi) = O(\beta), \qquad R(\beta, 2s\pi) = O(\beta^2),$$
$$I(\beta, (2s-1)\pi) = O(1), \qquad I(\beta, 2s\pi) = O(\beta).$$

The right-hand side of (159) is therefore equal to

$$n^2\pi^2 G_0(1 + 0.5\beta) + O(\beta^2),$$

and, applying (158) and remembering the restrictions (126) and (127), we have

$$8G_2 = n^2\pi^2 G_0(2 + \beta) + O(\beta^2). \tag{161}$$

That G_2 is positive means that F_i^{-2} increases with A^2, according to (127), or that c^2 decreases with A^2. This is a rather unexpected result, since we are so used to the increase of wave velocity with amplitude for progressive gravity waves, as for the well known Stokesian waves. We note, however, that already, for gravity waves in two superposed layers of homogeneous liquids, Hunt [1961] and Thorpe [1968a] (who made some corrections of Hunt's work) have shown that it is possible for c^2 to decrease with amplitude. (In their case it happens for long waves and small density differences.) The result (161) is remarkable in that for a continuously stratified fluid c^2 always decreases with the amplitude. Perhaps this result needs some interpretation. In this writer's opinion it happens because a larger amplitude in a wave motion "squeezes" the isopycnic lines against the upper solid boundary near the crests, thereby increasing the vertical density gradient near the upper solid boundary and decreasing the density gradient below. Similarly, the wave motion also "squeezes" the isopycnic lines against the lower solid boundary near the troughs, increasing the density gradient there and decreasing it above. Since the increase of density gradient near solid boundaries is quite ineffective in increasing c^2, whereas the decrease of it in a wider region is more effective in decreasing it, c^2 decreases with the amplitude.

The decrease of c^2 with amplitude can be understood in the simpler case of two superposed fluids between two horizontal plates. Let the mean depth of the lower fluid be small. For finite amplitudes of interfacial waves the minimum depth of the lower fluid can be considerably less than its mean depth. If c^2 does not decrease with the amplitude, when the amplitude is large the velocity in the lower fluid at its minimum depth, in a frame moving with the waves, would be so large that the pressure there would be very low—too low to maintain the equality of pressure for the fluids in contact, at the trough of the interface.

We have specified the stratification of ρ to be exponential by (124). However, the conclusions concerning the decrease of c^2 with A^2 for small β will remain valid for any stratification for which

$$\frac{1}{\rho}\frac{d\rho}{d\bar{\eta}} = \beta + O(\beta^2), \qquad \beta \ll 1. \tag{162}$$

In particular, they remain valid for the important case of a linear stratification, for which

$$\frac{d\rho}{d\bar{\eta}} = \text{constant}.$$

Whether β is small or not, and for any stratification, the method of solution presented here applies to finite-amplitude waves in any stratified fluid.

We note that the averaging process leading to (135) is valid only if there are no regions of closed streamlines. This requirement places a restriction on the admissible values of A^2. However, if A^2 is so large that regions of closed streamlines do occur, the modification required is obvious: All we have to do is to find how much fluid there is between any two streamlines (or isopycnic lines). The only thing that has to be kept in mind is that the solution must be such that when the fluid is allowed to quiet down it will have the stratification we specify for it.

14.1. *Discussions and Conclusions*

Solitary waves are classified as long waves, and they invariably owe their existence to the crucial requirement that the "local" wave velocity at the point of greatest vertical displacement be greater than the wave velocity of infinitesimal long waves. For a layer of homogeneous liquid, this requirement is satisfied [Lamb, 1945]. For two semi-infinite homogeneous liquids with an interface, Hunt [1961] has shown that this requirement is met [see Thorpe 1968a, p. 570]. For two homogeneous liquids of equal, finite depths bounded above and below by solid boundaries, whether this requirement is met or not depends on the wave length and the density difference [Hunt, 1961, as corrected by Thorpe, 1968a, p. 571, Eq. (2.1.6)]. This fact agrees with the conclusion of Long [1958].

Solitary waves in a continuously stratified fluid of finite depth, bounded above and below by horizontal boundaries have been treated elegantly by Benjamin [1966] and ingeniously by Long [1958]. In both papers it was assumed, for the existence of solitary waves, that the velocity of finite-amplitude waves increases with amplitude [see Benjamin, 1966, p. 243, first paragraph; Long, 1958, Eqs. (18), (42), and (66)]. The results of Section 9 show that, if solitary waves are indeed possible for weakly stratified fluids, their existence and the variation of wave velocity with wave amplitude for periodic waves are two entirely separate things altogether, and the increase of wave velocity with wave speed for solitary waves must be considered as an assumption verified *a posteriori* by the solitary-wave theory, unreachable by periodic waves with however large a wave length.

We can show, by the same argument as that applied to two superposed layers between horizontal plates to show that if one layer is very thin wave velocity must decrease with amplitude, that in such a two-layer system the speed of a solitary wave, if the interface were displaced toward the nearby plate, would decrease with amplitude. Hence the interface must be displaced away from the nearby plate. For stratified fluids, the solitary wave depicted in Fig. 2 of Thorpe [1968a] is possible only because the streamlines are displaced *away* from one boundary *more* pronouncedly than they are displaced toward the other boundary. It is also important to remember that the *distribution* of stratification is very important in determining the characteristics of solitary waves in a stratified fluid.

In the present context, the Boussinesq approximation amounts to ignoring the term $|\nabla\psi|^2$ and writing $\rho_0^{-1}\, d\rho/d\psi$ for $\rho^{-1}\, d\rho/d\psi$ in (11), ρ_0 being a constant mean density. Thus, the Boussinesq approximation does not necessarily make (11) linear. If we make the Boussinesq approximation in the present problem, Ψ_1 will be modified to the extent of a term of $O(\beta)$, and Ψ_2 will be modified. The term G_2 will be modified by an amount of at most $O(\beta)$ if k satisfies (152) or (153). Thus, the Boussinesq approximation will not affect the value of F_i^{-2} by an appreciable amount if β is small and k is not too large.

If we merely ignore the term $|\nabla\psi|^2$ in (11) and carry out the calculations for the exponentially stratified fluid, we shall find that Ψ_0 and Ψ_1 are unaffected, Ψ_2 is zero, and (161) always holds.

As shown in Section 3, the exactly linear cases of (11) can be found most easily from (10), by simply setting $d\rho/d\psi'$ and $h(\psi')$ to be linear functions of ψ'. The solutions for (10), when it is exactly linear, can then be obtained for any amplitude of the waves. The exactly linear cases all require a parallel flow of nonuniform velocity superposed on the wave motion. Granted this requirement, it is generally accepted that waves of any mode propagate with the same velocity in a stratified fluid regardless of the amplitude, permitting (10) to be exactly linear.

This, however, would be a somewhat superficial conclusion because the distribution of density in space varies with the amplitude, as we have shown, and the density stratification, if the waves were allowed to die out, would be different for different amplitudes of the waves. Hence, when we say the wave velocity is the same for any amplitude for the exactly linear cases, we are really comparing solutions not only for different wave amplitudes but for different fluids. This much neglected fact reduces the significance of the exactly linear cases.

On the assumption that the procedure of solution used in this paper is convergent, from the foregoing analysis and discussion we conclude the following.

1. Finite-amplitude waves of permanent form in any stably stratified fluid between two horizontal boundaries exist, and there are infinitely many modes. (The stability of these waves is another question.)

2. The method of solution presented in Section 14 can be applied to find the eigenfunction and the wave velocity for internal waves of finite amplitude in any stratified fluid.

3. In exceptional cases described by equations like (147) it is necessary to start with more than one wave number even at the first approximation.

4. For the first mode, calculations up to the order $O(A^2)$ for the eigenfunction and for the wave velocity (contained in F_i^2) how that for an exponentially stratified fluid and for a weak stratification the square of the wave velocity always decreases with the square of the amplitude provided (152) is satisfied, so that the wavelength is not too small. This conclusion is especially accurate when the amplitude is small, and holds for other weakly stratified fluids too.

5. Conclusion 4 is valid for higher modes also, provided (153) holds. Otherwise, further approximations are necessary to insure sufficient accuracy.

6. For weak stratifications and if the wavelengths are not very short, the Boussinesq approximation indeed gives reliable results for the wave velocity.

7. The solutions for different amplitudes for any of the exactly linear cases of (10) or (11) really correspond not to the same fluid, but to different fluids.

We note, before passing on to study solitary and cnoidal waves, that standing internal waves of finite amplitude have been studied in the excellent paper by Thorpe [1968b], who also presented beautiful photographs to support his theory.

15. INTERNAL SOLITARY AND CNOIDAL WAVES

In a paper remarkable for its elegance and originality, Benjamin [1966] gave results for solitary and cnoidal waves of finite amplitude and permanent form in a stratified incompressible fluid. The idea is rather simple, and can be simply represented. He started by defining the quantity

$$\int_A^B (p\,dz + \rho u\,d\psi) \tag{163}$$

(where ψ is the stream function) between two points A and B, which is independent of the path provided the flow is steady and there is no singularity between any two different paths, this independence being a consequence of the momentum principle. If we perform the integral at a vertical section between the bottom, where $z = 0$, and the upper boundary, where $z = h$, and evaluate p in terms of H given by (9), we can write the integral as

$$S = \int_0^h [H + \tfrac{1}{2}\rho(\psi_z^2 - \psi_x^2) - g\rho z]\,dz. \tag{164}$$

If we adopt a coordinate system traveling with the wave or waves to the left the flow is steady with respect to that system. In the case of the solitary wave the flow far upstream is parallel. In the case of cnoidal waves we can consider them to have been created by a moving barrier of special shapes, far downstream of which the cnoidal waves are in their pure form and far upstream of which the flow is parallel. In the parallel part of the flow the height of a particle will be denoted by η. [Note that this definition of η is different from that of the η in (18).] The elevation z of the same particle at other places will then depend on x. Thus for any particle

$$z = z(x, \eta). \tag{165}$$

The velocity $W(\eta)$ far upstream is $U(\eta) + c$, if $U(\eta)$ is the velocity with respect to fixed coordinates and c the wave speed. Obviously

$$\frac{d\psi}{d\eta} = W(\eta), \tag{166}$$

and

$$u = \psi_z = W(\eta)/z_\eta, \qquad w = -\psi_x = W(\eta)z_x/z_\eta, \tag{167}$$

since ψ is a function of η only, and $\partial\eta/\partial x$ for constant z is $-z_x/z_\eta$. Then, with Q denoting ρW^2,

$$S = \int_0^{h_0} \left[H + \frac{1}{2}Q\left(\frac{1 - z_x^2}{z_\eta^2}\right) - g\rho z\right] z_\eta \, d\eta \tag{168}$$

and far upstream

$$S_0 = \int_0^{h_0} \left(H + \frac{1}{2}Q - g\rho\eta\right) d\eta. \tag{169}$$

When wave motion is present we can write

$$z = \eta + \varepsilon\zeta(x, \eta), \tag{170}$$

in which ε is a small finite number. If we now evaluate $S - S_0$ in powers of ε, we have, to the first power,

$$\begin{aligned} S - S_0 &= \varepsilon \int_0^{h_0} \left[\left(H - \frac{1}{2}Q - g\rho\eta\right)\zeta_\eta - g\rho\zeta\right] d\eta \\ &= \varepsilon \int_0^{h_0} (p_0\zeta_\eta - g\rho\zeta)\, d\eta = \varepsilon[p_0\zeta]_0^{h_0}, \end{aligned} \tag{171}$$

since $dp_0/d\eta = -g\rho$. Since $\zeta = 0$ at the bottom and either $p_0 = 0$ or $\zeta = 0$ at the upper surface, $S - S_0 = 0$ to the first order in ε.

The Bernoulli equation at the free surface is

$$H(h_0) = \frac{1}{2}Q + g\rho h_0 = \frac{1}{2}Q\left(\frac{1 + z_x^2}{z_\eta^2}\right) + g\rho z. \tag{172}$$

Using (171) and (172), and linearizing in ε, we obtain

$$Q\zeta_\eta = g\rho\zeta \qquad \text{at} \quad \eta = h_0 \tag{173}$$

as the free-surface condition for infinitesimal waves. With this condition, we are ready to calculate $S - S_0$ to the second order in ε. To $O(\varepsilon^2)$, (168) and (169) give

$$\begin{aligned} S - S_0 &= \varepsilon^2 \int_0^{h_0} \left[\frac{1}{2} Q(\zeta_\eta^2 - \zeta_x^2) - g\rho\zeta\zeta_\eta\right] d\eta \\ &= -\frac{1}{2}\varepsilon^2 \int_0^{h_0} [Q\zeta_x^2 + (Q\zeta_\eta)_\eta\zeta - g\rho_\eta\zeta^2]\, d\eta, \end{aligned} \tag{174}$$

an integrated term

$$\frac{1}{2}\varepsilon^2[\zeta(Q\zeta_\eta - g\rho\zeta)]_0^{h_0}$$

having been dropped because it vanishes by virtue of the boundary conditions, whether the upper surface is rigid or free.

For solitary waves $S = S_0$. For periodic waves S is different from S_0 but is independent of x, since the bottom is flat in the part of the fluid where waves have been established. Differentiating (174) with respect to x, we have then, in any case,

$$0 = -\varepsilon^2 \int_0^{h_0} \zeta_x[Q\zeta_{xx} + (Q\zeta_\eta)_\eta - g\rho_\eta\zeta]\, d\eta \tag{175}$$

for all values of x in the region occupied by waves. Since ζ_x is not zero for all x, this suggests that

$$Q\zeta_{xx} + (Q\zeta_\eta)_\eta - g\rho_\eta\zeta = 0, \tag{176}$$

which is the equation governing infinitesimal two-dimensional waves. With this it can be shown that $S - S_0$, which is the wave resistance on the body creating the waves, is

$$S - S_0 = \frac{1}{2}\varepsilon^2 \int_0^{h_0} Q(\zeta_x^2 - \zeta\zeta_{xx})\, d\eta. \tag{177}$$

It is immediately evident that for periodic waves the wave resistance is always positive. It is also evident that if ζ_η and $\zeta_{\eta\eta}$ are of order 1 and, for long waves ζ_{xx} is of order ε, then $S - S_0$ is $O(\varepsilon^3)$.

To $O(\varepsilon)$,

$$\zeta(x, \eta) = \phi(\eta) \sin(\alpha x + v),$$

where

$$\frac{d}{d\eta}\left(Q\frac{d\phi}{d\eta}\right) - \left(Q\alpha^2 + g\frac{d\rho}{d\eta}\right)\phi = 0,$$

with boundary conditions

$$\phi(0) = 0$$

and

$$\phi(h_0) = 0 \quad \text{or} \quad Q\frac{d\phi}{d\eta} = g\rho\phi \qquad \text{at} \quad \eta = h_0.$$

according as the upper surface is fixed or free. The eigenvalues $c_1, c_2, \ldots,$ for this system are assumed to exist. The corresponding values for Q will be denoted by Q_1, Q_2, etc., which are still functions of η.

Benjamin used the new space variable $X = \varepsilon^{1/2}x$, so that

$$\zeta_x^2 = \varepsilon\zeta_X^2, \tag{178}$$

where ζ_X^2 is $O(1)$ for long waves. Furthermore, if the amplitude is no longer infinitesimal, Q should deviate from Q_n somewhat. Benjamin took

$$Q = Q_n + \varepsilon\gamma_n, \tag{179}$$

where γ_n is a function of η, and $\gamma_n > 0$ for waves of finite amplitude.

Equations (177) and (178) indicate that

$$S_0 - S = \varepsilon^3 s, \tag{180}$$

where s is $O(1)$. Benjamin even included a little loss of energy and assumed that

$$\int_0^{h_0} (H_0 - H)\, d\eta = \varepsilon^3 r. \tag{181}$$

where $r = O(1)$. This done, one can get from (168) and (169) the equation

$$\begin{aligned} 2\varepsilon^3(r - s) &= \int_0^{h_0} [\varepsilon^2(Q\zeta_\eta^2 - 2g\rho\zeta\zeta_\eta) - \varepsilon^3 Q(\zeta_X^2 + \zeta_\eta^3)]\, d\eta \\ &= \varepsilon^2[\zeta(Q\zeta_\eta - g\rho\zeta)]_0^{h_0} \\ &\quad - \int_0^{h_0} \{\varepsilon^2[(Q\zeta_\eta)_\eta - g\rho_\eta\zeta]\zeta + \varepsilon^3 Q(\zeta_X^2 + \zeta_\eta^3)\}\, d\eta. \end{aligned} \tag{182}$$

Introducing (179) and

$$\zeta = f(X)\phi_n(\eta), \tag{183}$$

and noting that $\phi_n(\eta)$ satisfies the boundary conditions at the lower and upper surface, Benjamin obtained, upon collecting terms of $O(\varepsilon^3)$,

$$\begin{aligned} 2(r - s) &= f^2\left[\gamma_n\phi_n\frac{d\phi_n}{d\eta}\right]_0^{h_0} - f^2\int_0^{h_0}\phi_n\frac{d}{d\eta}\left(\gamma_n\frac{d\phi_n}{d\eta}\right)d\eta \\ &\quad - f_X^2\int_0^{h_0} Q\phi_n^2\, d\eta - f^3\int_0^{h_0} Q_n\left(\frac{d\phi_n}{d\eta}\right)^3 d\eta, \end{aligned} \tag{184}$$

which, after integrating by parts the second term on the right-hand side, becomes

$$If_\chi^2 = Jf^2 - Kf^3 + 2(s - r), \tag{185}$$

where

$$I = \int_0^{h_0} Q\phi_n^2 \, d\eta, \qquad J = \int_0^{h_0} \gamma_n\left(\frac{d\phi_n}{d\eta}\right)^2 d\eta, \qquad K = \int_0^{h_0} Q_n\left(\frac{d\phi_n}{d\eta}\right)^3 d\eta. \tag{186}$$

Equation (185) is the Korteweg-DeVries equation, from which solutions for solitary and cnoidal waves can be obtained for all values of n, that is, for all modes of these waves. This finishes the structural description of Benjamin's theory.

Benjamin noted that K is $O(1)$ when $n = 0$, but $O(\beta^2)$ for $n = 1$, β being a measure of the density gradient. (The demonstration that K is $O(\beta^2)$ for $n = 1$ is a little involved. We refer the reader to the appendix of Benjamin's paper.) The approximation to the order ε^3 is valid only if

$$K \gg \varepsilon L, \qquad \text{where} \quad L = \int_0^{h_0} Q_n\left(\frac{d\phi_n}{d\eta}\right)^4 d\eta = O(1).$$

Thus it appears at first sight that for internal modes the theory is adequate only if $\beta^2 \gg \varepsilon$, which would impose a severe restriction on the usefulness of the theory. Benjamin suggested the "simple expedient" of using $\phi_n(z) = \phi_n(\eta + \varepsilon\zeta)$ in (183) through (186), which removes the restriction. That $\phi_n(z)$ is more suitable than $\phi_n(\eta)$ is perhaps acceptable on intuitive grounds, for the eigenfunctions should reflect the local conditions more than the conditions far upstream. Nevertheless, the expedient of using $\phi_n(z)$ instead of $\phi_n(\eta)$ lacks the quality of inevitability, and a formal justification would have been welcome.

Since the determination of γ_n is left unspecified, it is desirable to see how a solution can be obtained without *first* determining it. Take the case of the solitary wave in a homogeneous fluid with $U = 0$, for which $s = 0$, since no obstacle is needed to create the wave and the constant S is just the S_0 far upstream. If energy loss is neglected $r = 0$, and a solution for the solitary wave exists for $J > 0$:

$$f = a \operatorname{sech}^2 \kappa X, \qquad a = J/K, \qquad \kappa = \frac{1}{2}(J/I)^{1/2},$$

Hence

$$z - \eta = \varepsilon\zeta = \frac{\varepsilon J\phi_n}{K} \operatorname{sech}^2\left[\frac{x}{2}\left(\frac{\varepsilon J}{I}\right)^{1/2}\right] \tag{187}$$

Here $\phi_0 = \eta$, and

$$I = \frac{1}{3}\rho c^2 h_0^3, \qquad \varepsilon J = \rho(c^2 - c_0^2)h_0, \qquad K = \rho c_0^2 h_0,$$

in which ρ is the constant density, $c_0^2 = gh_0$ (c_0 = speed of infinitesimal waves), and c is the speed of the solitary wave. Equation (187) can be written as

$$z - \eta = \eta\left(\frac{c^2}{c_0^2} - 1\right)\operatorname{sech}^2\left[\frac{(3c^2 - 3c_0^2)^{1/2}x}{2ch_0}\right].$$

For $\eta = 1$ (top streamline),

$$z(h_0) - h_0 = \Delta \operatorname{sech}^2\left[\left(\frac{3\Delta}{h_0 + \Delta}\right)^{1/2}\frac{x}{h_0}\right],$$

in which $c^2 = g(h_0 + \Delta)$. Thus c^2, or J or γ_0, is related to the amplitude of the solitary wave, as well as its shape.

Solitary waves in an incompressible fluid with an exponentially distributed density have been investigated by Long [1965], who found that the relative magnitudes of three small parameters are very important. These parameters are the amplitude, the reciprocal of the effective length of the wave, and the density gradient. The explicit examples given by Long indicate the danger of using the Boussinesq approximation when other small parameters (than the density variation) exist. His conclusion is supported by Benjamin's work [1966].

It is appropriate to add here that time-dependent equations for the development of weakly nonlinear waves in a stratified shear flow have been derived by Benney [1966], who also gave solutions of these equations. Solitary waves in a compressible, stratified fluid confined between two rigid boundaries and flowing with uniform velocity at infinity have been studied by Long and Morton [1965]. Shen [1966] studied solitary waves in a compressible stratified fluid of infinite height and with arbitrary wind and density profiles, and later [1967] gave results for long waves (solitary and cnoidal) of finite amplitude in an atmosphere with crosswind.

It is also pertinent to mention here the work of Benjamin [1967] on internal long waves of permanent form in a stratified fluid layer with an overlying or underlying homogeneous fluid or infinite depth. The crucial point is that while the waves are long with respect to the stratified layer they are, of course, not long with respect to the fluid of infinite depth, and the usual analysis for long waves has to be modified in such a way that the long-wave solution and the solution for the homogeneous fluid can be matched at the interface (where Benjamin assumed no density discontinuity). Benjamin started by noting that, for linear waves in two layers of homogeneous fluids of depths h_1 and h_2, the dispersion equation for long waves are radically different for the two cases h_2 finite and h_2 infinite, if h_1 is kept finite, and based the subsequent work on this observation. Space does not allow us to dwell long here, except to mention that solitary waves of the form

$$\text{vertical displacement of the interface} = \frac{a\lambda^2}{x^2 + \lambda^2}$$

are obtained, where a and λ are constants depending on the wave amplitude and the depth of the stratified layer.

16. STEADY FLOWS OF A STRATIFIED FLUID IN THREE DIMENSIONS

Yih [1967b] derived the equations governing steady three-dimensional flows of a stratified, inviscid, and nondiffusive fluid. For this derivation, it is convenient to write Eqs. (1.9) and (1.21) in their vector forms, with their time-derivative terms dropped,

$$(\mathbf{v} \cdot \mathbf{grad})\, \rho = 0 \qquad \text{or} \qquad (\mathbf{v} \cdot \mathbf{grad})\, S = 0, \tag{188}$$

in which $\mathbf{v}$ is the velocity vector, ρ the density, and S the entropy. These equations show that in steady flows the velocity vector must lie in isopycnic surfaces or surfaces of constant entropy. Since the circulation along any circuit lying entirely in an isopycnic surface or a surface of constant entropy is preserved, this circulation must be zero if the flow is supposed to have been established from rest. Hence, for such a flow the vorticity vector must also lie in an isopycnic surface or a surface of constant entropy. (See Section 5, Chapter 1.) Then for the case of an incompressible fluid

$$\mathbf{v} = \mathbf{grad}\, a \times \mathbf{grad}\, \rho, \tag{189a}$$

$$\boldsymbol{\omega} = \mathbf{grad}\, b \times \mathbf{grad}\, \rho, \tag{189b}$$

in which $\boldsymbol{\omega}$ is the vorticity vector, a is a stream function [Yih, 1957b], the other stream function being ρ, and b is the vorticity function of Clebsch [Lamb, 1945, p. 248], the other vorticity function being ρ. Of course $\mathbf{v}$ and $\boldsymbol{\omega}$ are related by

$$\boldsymbol{\omega} = \mathbf{curl}\, \mathbf{v}. \tag{190}$$

For a compressible fluid (189a) and (189b) are to be replaced by

$$\rho\mathbf{v} = \mathbf{grad}\, a \times \mathbf{grad}\, S \tag{191a}$$

$$\boldsymbol{\omega} = \mathbf{grad}\, b \times \mathbf{grad}\, S. \tag{191b}$$

16.1. *Derivation of the Equations for an Incompressible Fluid*

The equations of steady motion are

$$(\rho\mathbf{v} \cdot \mathbf{grad})\, \mathbf{v} = -\mathbf{grad}\, p + \rho\mathbf{F}, \tag{192}$$

in which p is the pressure and $\mathbf{F}$ the body force per unit mass, given by

$$\mathbf{F} = -\mathbf{grad}\, gz,$$

z being the Cartesian coordinate measured in the direction of the vertical. If we assume [Yih, 1958]

$$\mathbf{v}' = (\rho/\rho_0)^{1/2}\mathbf{v}, \qquad \boldsymbol{\omega}' = \mathbf{curl}\, \mathbf{v}', \tag{193}$$

(191) can be written as

$$(\rho_0 \mathbf{v}' \cdot \nabla)\, \mathbf{v}' = -\mathbf{grad}\, p + \rho\mathbf{F}, \tag{192a}$$

which can further be written as

$$-\rho_0 \mathbf{v}' \times \boldsymbol{\omega}' = -\mathbf{grad}\, \chi + \rho\mathbf{F}, \tag{194}$$

with

$$\chi = p + \frac{1}{2}\rho(u^2 + v^2 + w^2),$$

u, v, and w being the components of $\mathbf{v}$ in the directions of increasing Cartesian coordinates x, y, and z, respectively, and ρ_0 being a reference density.

Now since $\mathbf{v}'$ and $\boldsymbol{\omega}'$ are still solenoidal, we can write

$$\mathbf{v}' = \mathbf{grad}\, \alpha \times \mathbf{grad}\, \rho, \tag{195a}$$

$$\boldsymbol{\omega}' = \mathbf{grad}\, \beta \times \mathbf{grad}\, \rho. \tag{195b}$$

Then a simple calculation shows that

$$\mathbf{v}' \times \boldsymbol{\omega}' = -(\mathbf{v}' \cdot \mathbf{grad}\, \beta)\, \mathbf{grad}\, \rho = (\boldsymbol{\omega}' \cdot \mathbf{grad}\, \alpha)\ \mathbf{grad}\, \rho. \tag{196}$$

This is as it should be, since both the velocity vector and the vorticity vector lie in a surface of constant ρ, and thus their vector product must be parallel to **grad** ρ. Substituting (196) in (194), separating the result into three equations, multiplying these respectively by dx, dy, and dz, and addding, we obtain, after an obvious simplification.

$$\rho_0 \boldsymbol{\omega}' \cdot \mathbf{grad}\, \alpha = (dH/d\rho) - gz, \tag{197}$$

in which

$$H = \chi + \rho g z \tag{198}$$

is a function of ρ only, since the Bernoulli function is constant in an isopycnic surface for steady flows. Now, since $\boldsymbol{\omega}'$ is in an isopycnic surface,

$$\boldsymbol{\omega}' \cdot \mathbf{grad}\, \rho = 0. \tag{199}$$

We shall now use the second equation in (193) and (195a) to obtain

$$\begin{aligned} \xi' &= (\rho_{yy} + \rho_{zz})\alpha_x - \rho_{xy}\alpha_y - \rho_{xz}\alpha_z - (\alpha_{yy} + \alpha_{zz})\rho_x + \alpha_{xy}\rho_y + \alpha_{xz}\rho_z, \\ \eta' &= (\rho_{zz} + \rho_{xx})\alpha_y - \rho_{yx}\alpha_x - \rho_{yz}\alpha_z - (\alpha_{zz} + \alpha_{xx})\rho_y + \alpha_{yx}\rho_x + \alpha_{yz}\rho_z, \\ \zeta' &= (\rho_{xx} + \rho_{yy})\alpha_z - \rho_{zx}\alpha_x - \rho_{zy}\alpha_y - (\alpha_{xx} + \alpha_{yy})\rho_z + \alpha_{zx}\rho_x + \alpha_{zy}\rho_y, \end{aligned} \tag{200}$$

in which ξ', η', and ζ' are the three components of $\boldsymbol{\omega}'$. With (200) introduced in (197) and (199), we obtain two equations of the two unknowns α and ρ, which are the equations sought. For two-dimensional flows in the (x, z)-plane, $\boldsymbol{\omega}'$ has only the component η' and **grad** ρ is in the (x, z)-plane. Hence (199) is automatically satisfied. Furthermore, if we keep the dimensions correct and write

$$\alpha = (Vd/\rho_0)y,$$

in which V is a reference velocity, d a reference length, and ρ_0 a reference density, we have

$$\rho_{xx} + \rho_{zz} = \frac{\rho_0}{(Vd)^2}\left(\frac{dH}{d\rho} - gz\right). \tag{201}$$

With

$$\xi = \frac{x}{d}, \qquad \zeta = \frac{z}{d}, \qquad r = \frac{\rho}{\rho_0}, \qquad h = \frac{H}{\rho_0 V^2}, \qquad F^2 = \frac{V^2}{gd},$$

(201) becomes

$$r_{\xi\xi} + r_{\zeta\zeta} = (dh/dr) - F^{-2}\zeta. \tag{202}$$

We can also recover Yih's form of the equation of Dubreil-Jacotin if we put

$$\alpha = (d\psi'/d\rho)y,$$

in which ψ' is the modified stream function used in (2). Since neither ψ' nor ρ depends on y, the result is (10), the equation of Dubreil-Jacotin in Yih's form.

Equations (197) and (199), with $\boldsymbol{\omega}'$ given by (200), are integrated forms of the equations of motion. That (197) results from integration is obvious from its derivation. Even (199) results from an integration—the integration along any closed circuit in an isopycnic line to obtain the circulation, which is zero if the motion started from rest. Thus (197) and (199) are several steps in advance of the Euler equations of motion. The equation of continuity is automatically satisfied by (189a) or (195a). The first equation in (188), which has been used to obtain (192a), is also automatically satisfied by (189a), or (195a) taken together with the definition of $\mathbf{v}'$ in (193).

The left-hand side of (197) is linear in ρ, and the left-hand side of (199) is linear in α. In this sense the left-hand sides of (197) and (199) are quasi-linear. If $dH/d\rho$ is linear in ρ, (197) and (199) are quasi-linear, and a calculation can be performed by first assuming a plausible form for α, solving (197) for ρ, using the result in (199) and solving it for α, and repeating the process.

16.2. *Derivation of the Equations for a Compressible Fluid*

For a compressible fluid in steady motion the basic equations are still (192). The equation of continuity

$$\mathbf{div}\ \rho\mathbf{v} = 0$$

is automatically satisfied by (191a). We shall again use the variable λ defined by (1.25). With the substitutions (1.27) and $\mathbf{v} = (u_1, u_2, u_3)$, the basic equations of motion can be written as

$$(\rho'\mathbf{v}' \cdot \nabla)\,\mathbf{v}' = -\mathbf{grad}\ p' + \rho'\lambda\mathbf{F}, \tag{203}$$

in which

$$\rho'/(p')^{1/\gamma} = \text{constant.}$$

Equation (203) can be written as

$$-\mathbf{v}' \times \boldsymbol{\omega}' = -\mathbf{grad}\ \chi + \lambda\mathbf{F}, \tag{204}$$

in which now

$$\chi = \int \frac{dp'}{\rho'} + \frac{q'^2}{2},$$

q' being the magnitude of $\mathbf{v}'$. The equation of continuity in terms of ρ' and $\mathbf{v}'$ is, in virtue of the second equation in (188),

$$\mathbf{div}\ \rho'\mathbf{v}' = 0,$$

which is automatically satisfied by

$$\rho'\mathbf{v}' = \mathbf{grad}\ \alpha \times \mathbf{grad}\ \lambda.$$

We use this form not only because the satisfaction of the equation of continuity is assured, but also because the velocity vectory must lie in surfaces of constant S or λ. Similarly, since the vorticity vector ω (and hence ω') must also lie in surfaces of constant λ,

$$\boldsymbol{\omega} = \mathbf{grad}\ \beta \times \mathbf{grad}\ \lambda.$$

A simple calculation shows that

$$\rho'\mathbf{v}' \times \boldsymbol{\omega}' = -(\rho'\mathbf{v}' \cdot \mathbf{grad}\ \beta)\ \mathbf{grad}\ \lambda = (\boldsymbol{\omega}' \cdot \mathbf{grad}\ \alpha)\ \mathbf{grad}\ \lambda.$$

Substituting this in (203), multiplying it by dx, and integrating, we obtain

$$\frac{1}{\rho'}\,\boldsymbol{\omega}' \cdot \mathbf{grad}\ \alpha = \frac{dH}{d\lambda} - gz, \tag{205}$$

in which now H is given by (91). The other equation is

$$\boldsymbol{\omega}' \cdot \mathbf{grad}\ \lambda = 0. \tag{206}$$

The expression for $\boldsymbol{\omega}'$ is now a little more complicated. It is

$$\boldsymbol{\omega}' = \textbf{curl}\ \mathbf{v}' = \textbf{curl}\ [(1/\rho')\ \textbf{grad}\ \alpha \times \textbf{grad}\ \lambda]. \tag{207}$$

We shall not expand it in full. Equation (205) with $\boldsymbol{\omega}'$ given by (206) is the vector equation sought. The quantity ρ' can be expressed in terms of H, q', and λ by the use of (91).

For two-dimensional flows,

$$\alpha = \rho_0 V y d,$$

and (205) becomes

$$\left(\frac{\rho_0 V d}{\rho'}\right)^2 \left[(\lambda_{xx} + \lambda_{zz}) - \frac{1}{\rho'}(\lambda_x \rho'_x + \lambda_z \rho'_z)\right] = \frac{dH}{d\lambda} - qz.$$

If we put

$$\alpha = (d\psi'/d\lambda)y,$$

we obtain (93), Yih's form of the equation of Dubreil-Jacotin.

Equation (205) is a few steps in advance of the basic equations of motion because they have been obtained from the latter equations by integration.

Actually (197) and (199), governing the motion of a stratified incompressible fluid, are also the equations governing steady vortex motion of a homogeneous incompressible fluid. Isopycnic surfaces would of course have no definite meaning, but we can replace ρ by L, which is constant on a Lamb surface with streamlines and vorticity lines imbedded in it. Since ρ is now constant, the last term in (197) drops out for vortex motion, which is then governed by

$$\rho\boldsymbol{\omega} \cdot \textbf{grad}\ \alpha = \frac{dH}{dL} \quad \text{and} \quad \boldsymbol{\omega} \cdot \textbf{grad}\ L = 0, \tag{208}$$

with $\boldsymbol{\omega}$ given by (200), in which ρ is replaced by L, and the accents on ξ', η', and ζ' are removed. Of course, the motion is now not assumed to have started from rest. It would be irrotational in that case.

For a homentropic gas in steady vortex motion, the governing equations are

$$\frac{1}{\rho}\boldsymbol{\omega} \cdot \textbf{grad}\ \alpha = \frac{dH}{dL} \quad \text{and} \quad \boldsymbol{\omega} \cdot \textbf{grad}\ L = 0, \tag{209}$$

in which

$$\boldsymbol{\omega} = \textbf{curl}\ \mathbf{v} = \textbf{curl}\ [(1/\rho)\ \textbf{grad}\ \alpha \times \textbf{grad}\ L]. \tag{210}$$

Equations (208) and (209) are the results of first integrations of the vorticity equations. It is a little surprising that they have not been found before.

Finally, we remark that the solution of (197) and (199), or (205) and (206), or (208), or (209), is not unique. For if α is a solution so is $\alpha + F(\rho)$, or $\alpha + F(\lambda)$, or $\alpha + F(L)$, as the case may be, but the velocity field is uniquely determined.

17. SHALLOW-WATER THEORY FOR STEADY STRATIFIED FLOWS IN THREE DIMENSIONS

The equations governing large-amplitude three-dimensional steady flows of a stratified fluid just presented are so nonlinear and complex that not a single solution for a truly three-dimensional case is known. If, however, the vertical scale of a stratified liquid is small compared with a representative horizontal scale, the pressure distribution at any section is essentially hydrostatic. As a consequence, the number of the spatial variables can be reduced from three to two, although the flow treated is still truly three-dimensional. The theory built on the basic assumption of small vertical scale is the so-called shallow-water theory. Yih [1969a] showed that, whenever the shallow-water theory assumption is valid, a class of exact solutions exists for steady flows of a stratified fluid. The principal result is that to any solution by the shallow-water theory for a steady flow of a homogeneous fluid there corresponds a solution for a steady flow of a stratified fluid with arbitrary stratification, the velocity field for the latter being obtained from that for the former by a transformation explicitly dependent on the density stratification, and that steady stratified flows issuing from a large reservoir enjoy this correspondence, provided the assumptions underlying the shallow-water theory are satisfied and the downstream conditions allow it.

We restrict our attention to steady flows of an incompressible, inviscid, and nondiffusive fluid of variable density. For such flows the equations of motion are (1.36). The equation of continuity is (1.10) and the equation of incompressibility is (1.17), with u_1, u_2, and u_3 replaced by u, v, and w.

We shall consider flows of which u and v are of the order of a representative velocity V_0, and the representative horizontal length is L. If we denote by ζ the vertical displacement of a fluid particle from its upstream elevation or mean elevation, then, since steady flows are considered,

$$u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = w. \tag{211}$$

The basic assumption of the shallow-water theory is

(i) $$(h/L)^2 = \varepsilon^2 \ll 1,$$

in which h is the depth of the fluid and a function of x and y. Since $\zeta < h$, (i) implies that $(\zeta/L)^2 < \varepsilon^2 \ll 1$. We shall assume that the bottom is flat and situated at $z = 0$. Integration of (1.10) with respect to z then produces the result

$$w = O(\varepsilon V_0), \tag{212}$$

in which V_0 is a representative velocity. Substituting (212) into the third equation in (1.36) and ignoring quantities of order ε or of higher order in ε, we then obtain

$$\frac{\partial p}{\partial z} = -g\rho \qquad \text{or} \qquad p = \int_z^h g\rho\, dz, \tag{213}$$

if the free surface is given by

$$z = h(x, y). \tag{214}$$

(We assume in this section that a free surface is present.) Equation (213) is the principal consequence of (i).

For the development of the shallow-water theory for stratified fluids in steady flow, a presentation of the shallow-water theory for a homogeneous liquid is essential. Consider a homogeneous fluid with a free surface flowing above a horizontal bed. The depth will be denoted by h. If we assume the upstream flow to be irrotational, or, more generally, the flow to have been started from rest, then the whole flow is irrotational, since the fluid is inviscid and the density constant. Since (212) is still valid under the basic assumption (i), the equations of irrotationality are, if U and V denote u and v for homogeneous fluids,

$$\frac{\partial V}{\partial z} = 0, \qquad \frac{\partial U}{\partial z} = 0, \qquad \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0 \tag{215}$$

if quantifies of order ε are neglected. The first two equations in (215) state simply that U and V are independent of z and the third allows the use of a velocity potential Φ, in terms of which

$$U = \Phi_x, \qquad V = \Phi_y, \tag{216}$$

in which subscripts denote partial differentiation. The equation of continuity is then, as can be shown in the usual way by taking as the control surface and the surface formed by the bottom, the free surface and the lateral surface of a vertical prism of cross section $dx\, dy$.

$$\frac{\partial(Uh)}{\partial x} + \frac{\partial(Vh)}{\partial y} = 0. \tag{217}$$

The Bernoulli equation is, with h_0 denoting the depth at a stagnation point or in a large reservoir,

$$U^2 + V^2 + 2gh = 2gh_0, \tag{218}$$

since the neglected term w^2 is of the order of ε^2. Instead of the equations of motion, we can simply use (218), which can be derived from them. Substitution of (218) into (217) produces

$$(c^2 - U^2)U_x - UV(U_y + V_x) + (c^2 - V^2)V_y = 0, \tag{219}$$

in which

$$c^2 = gh. \tag{220}$$

In virtue of (216), (219) can be written as

$$(c^2 - \Phi_x^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (c^2 - \Phi_y^2)\Phi_{yy} = 0, \tag{221}$$

The c^2 in (219) and (221) can be expressed in terms of U and V by the use of (218), or in terms of Φ in the further use of (216). It was Riabouchinsky [1932] who first pointed out the analogy between (221) and the equation governing the velocity potential of two-dimensional irrotational flows of a homentropic inviscid gas. Equation (221) is, as Riabouchinsky pointed out, identical to the equation governing two-dimensional irrotational motion of a homentropic gas obeying the law for isentropic change of state

$$p/\rho^\gamma = \text{constant}, \qquad \text{with} \quad \gamma = 2.$$

The equation in gas dynamics corresponding to (221) has been studied by Molenbroek [1890] and more fruitfully by Chaplygin [1904], both of whom used hodographic variables as independent variables.

Now, for the motion of a stratified liquid started from rest, vorticity will be created. But the vortex lines will lie in surfaces of constant density, so that the vorticity component normal to a surface of constant density is zero, as a direct consequence of the Kelvin theorem [see Section 5, Chapter 1]. Remembering assumption (i), this means that, with terms of order ε neglected, the third equation in (215) still stands. Thus we have irrotationality in a constant-density surface when the motion is viewed from above. This does not save (216) for the whole field of flow, but does save it for a constant-density surface. As to the first two equations in (215), they are certainly no longer valid.

We shall now show that, if the shallow-water assumption is satisfied, steady flow of a stratified fluid with a free surface issuing from a large reservoir can, although it does not necessarily, have a flow pattern exactly like that of a homogeneous fluid with a free surface, issuing from the same reservoir into the same channel. Since a flow having such a pattern is far from the only kind of flow a stratified fluid can have, it is sufficient to show that such a flow is dynamically permissible, i.e., it is consistent with the only two equations governing the flow: the equation of continuity and the Bernoulli equation.

We shall, then, assume that for every constant-density surface

$$\zeta/h = \zeta_0/h_0, \tag{222}$$

in which h_0 is the depth far upstream (in the reservoir), ζ_0 is the reservoir elevation of the constant-density surface, which has the elevation $\zeta(x, y)$ at other places, and $h(x, y)$ the depth at any (x, y). The pressure at any point is, under the shallow-water assumption,

$$p = g \int_z^h \rho(z')\, dz'. \tag{223}$$

At any fixed values of x and y, z' can be identified with ζ and dz' with $d\zeta$, and a change of the value of ζ involves a change of the value of ρ. Hence (223) can be written as

$$p = g \int_z^h \rho(\zeta)\, d\zeta = \frac{gh}{h_0} \int_{z_0}^{h_0} \rho_0(\zeta_0)\, d\zeta_0 = \frac{gh}{h_0} \int_{z_0}^{h_0} \rho_0(z_0')\, dz_0', \tag{224}$$

in which the validity of the second equality sign depends on (222).

Whether or not (222) is assumed, the Bernoulli equation written for any point (x, y, z) and a point far upstream on the same constant-density surface is

$$u^2 + v^2 + 2g\left(z + \frac{1}{\rho(z)} \int_z^h \rho(z')\, dz'\right) = C(\rho), \tag{225}$$

in which

$$C(\rho) = 2g\left(z_0 + \frac{1}{\rho_0(z_0)} \int_{z_0}^{h_0} \rho_0(z_0')\, dz_0'\right). \tag{226}$$

If (222) is assumed, (224) and (226) permit (225) to be written as

$$u^2 + v^2 + 2ghB(\rho) = 2gh_0 B(\rho), \tag{227}$$

in which

$$B(\rho) = C(\rho)/2gh_0. \tag{228}$$

If we now write

$$(u, v) = \lambda(\rho)(U, V), \qquad \lambda^2 = B(\rho), \tag{229}$$

(227) becomes

$$U^2 + V^2 + 2gh = 2gh_0, \tag{230}$$

the same as (218). That is, if the Bernoulli equation is satisfied by the flow of a homogeneous fluid, it is also satisfied by the stratified flow with the same flow pattern and a velocity distribution given by (229).

Note that, for steady flows, (211) gives

$$W = U \frac{\partial \zeta}{\partial x} + V \frac{\partial \zeta}{\partial y} \tag{231}$$

for a homogeneous fluid and

$$w = u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} \tag{232}$$

for a stratified fluid. Hence (229) also implies

$$w = \lambda(\rho) W. \tag{233}$$

Therefore, the equation of continuity is satisfied if

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0, \tag{234}$$

provided u and v are given by (229). The consistency of (222) and (229) with the equation of continuity and with the dynamical equations is thus established, and we state the

Theorem. *So long as the shallow-water theory is valid, a class of steady stratified flows with a free surface originated from rest can be found corresponding to each irrotational steady free-surface flow of a homogeneous fluid originated from rest. The mapping is by the use of* (229).

Note that even in the presence of a stagnant layer of fluid the flowing part of the stream can still obey the theorem. In other words, the theorem is true wherever the basic assumptions of the shallow-water theory are fulfilled and a free surface or stagnant upper layer is present. We shall now present a few examples. Examples 3 and 4 illustrate flows with a stagnant layer.

Example 1. Gravity jets of a stratified fluid. With reference to Fig. 30, the water level A behind the vertical wall far from the opening is higher than the water level in front of the vertical wall, which is flat except in the jet issuing from the opening. The curved free surface of the jet is higher than the flat level (straight part of B—B) of the dead water surrounding it, but approaches that level very far downstream. The bottom is horizontal throughout and the depth of water is assumed to be small compared with the opening in the wall. The problem for a homentropic gas was solved by Chaplygin [1904], and Ferguson and Lighthill [1947] calculated the coëfficient of contraction for $\gamma = 1{\cdot}4$. In the shallow-water case $\gamma = 2$, whereas for the classical Kirchhoff jet $\gamma = \infty$. The coefficient of contraction C_c is the ratio of the asymptotic width of the jet to the opening of the wall. For the Kirchhoff jet

$$C_c = \frac{\pi}{\pi + 2} = 0.611.$$

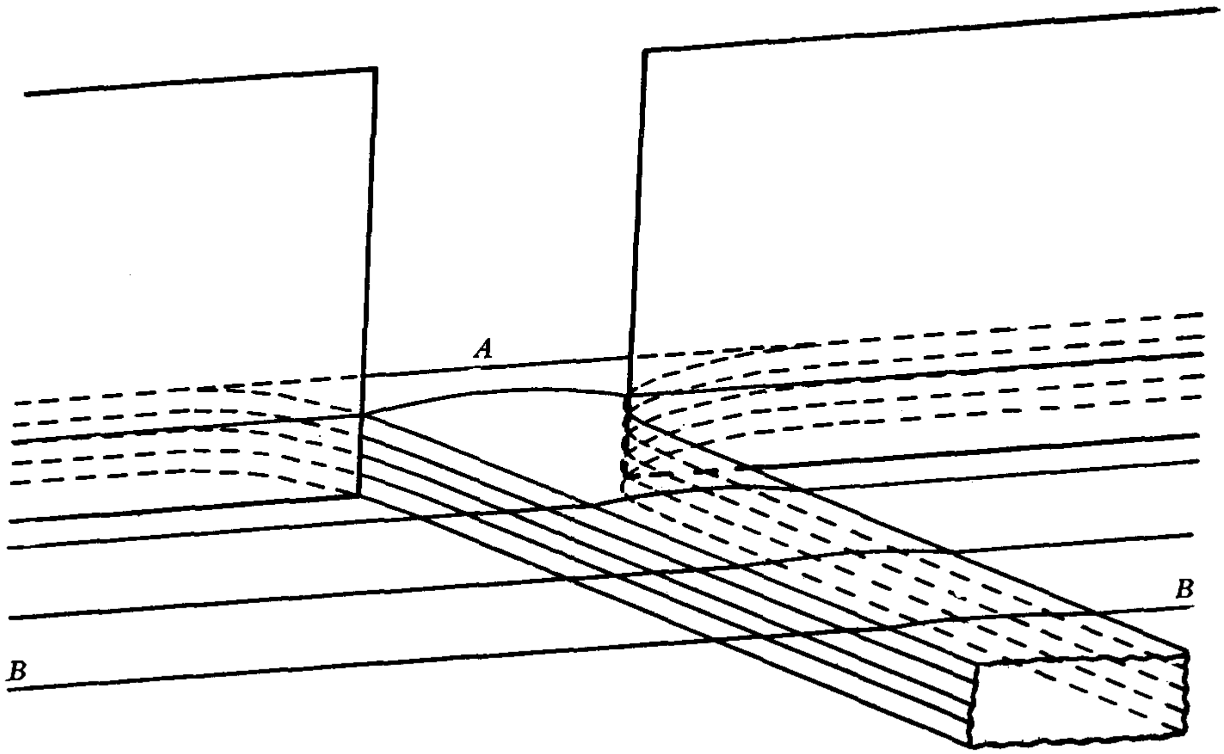


FIGURE 30. A perspective view of the gravity jet. The line A indicates the level of the free surface far upstream. The flat part of the intersection $B-B$ of the free surface with a vertical plane is on the surface of the stagnant liquid surrounding the jet. There are three other lines like $B-B$ in the figure. Their flat parts are at the same level as the flat part of $B-B$. The fluid may be homogeneous or stratified. (*Courtesy of J. Fluid Mech. and the Cambridge Univ. Press.*)

For the Chaplygin jet Ferguson and Lighthill [1947] gave C_c for $\gamma = 1.4$ and various values of

$$\tau_1 = \frac{q^2}{q_{\max}^2}, \tag{235}$$

q being the speed along the boundary of the jet and $q_{\max}$ the maximum speed attainable by the gas. In our case $\gamma = 2$ and $q_{\max}$ is now the maximum speed attainable by the liquid.

A calculation by Dr. C. H. Li, for $\gamma = 2$, is given in the accompanying table.

τ_1	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	—
C_c	$\pi/(\pi+2)$	0.6156	0.6205	0.6255	0.6307	0.6363	0.6419	0.6479	—
τ_1	0.16	0.20	0.22	0.24	0.26	0.28	0.30	0.32	$\frac{1}{3}$
C_c	0.6542	0.6677	0.6749	0.6825	0.6904	0.6987	0.7075	0.7167	0.7230

The maximum value of τ_1 for subcritical flow is $(\gamma - 1)/(\gamma + 1) = \frac{1}{3}$ for $\gamma = 2$.

For a stratified fluid with any stratification, we need (229) to obtain the velocity distribution. But the coefficient of contraction is the same if the flow pattern remains unchanged. It is tacitly assumed that, if the density far upstream is given by $\rho = f(z)$, that in the stagnant liquid surrounding the jet is given by $\rho = f(rz)$, with r equal to the ratio of the upstream depth to the depth far downstream, if the flow pattern is to remain the same as for a homogeneous liquid. This can be achieved by having two large basins divided by the wall, filling them while keeping the sluice gate open, and then closing the gate and enlarging in any way the area of the downstream basin, thus lowering the levels of the constant-density surfaces proportionally. When the gate is then opened, the condition at the edge of the jet is just what is needed for the solution to be physically relevant.

Example 2. Stratified flow in a channel expansion. Figures 31a and b show the plan and elevation (at the center plane) views of a homogeneous liquid flowing through a channel supercritically, i.e., with the velocity everywhere greater than the local speed of long waves of the gravest mode. Equation (221) is now entirely hyperbolic and the solution by the use of the method of characteristics is well known. For a stratified liquid with any stratification, again (229) provided the corresponding solution.

Example 3. Gravity jets with an overlying stagnant layer. In Fig. 32, if the flowing layer is homogeneous and has the constant density ρ_t, the gravity jet will be identical to the gravity jet without an overlying layer in every respect, except that the velocity is reduced by the factor $(\rho_t - \rho')/\rho_t$, ρ' being the density of the overlying layer. This can be easily seen, since the Bernoulli equation is now

$$U'^2 + V'^2 + 2g'h = 2g'h_0, \tag{236}$$

in which

$$g' = \frac{\rho_t - \rho'}{\rho_t}\, g, \tag{237}$$

and the primes on U and V are to indicate the presence of the overlying layer, for the sake of distinction.

If the flowing layer is stratified, the velocity distribution in a dynamically possible flow with the same flow pattern is given by

$$u = \lambda'(\rho)U', \qquad v = \lambda'(\rho)V', \tag{238}$$

in which ρ now varies from one surface to another and the prime on λ does not indicate differentiation. It is important to note that $\lambda'(\rho)$ is not simply

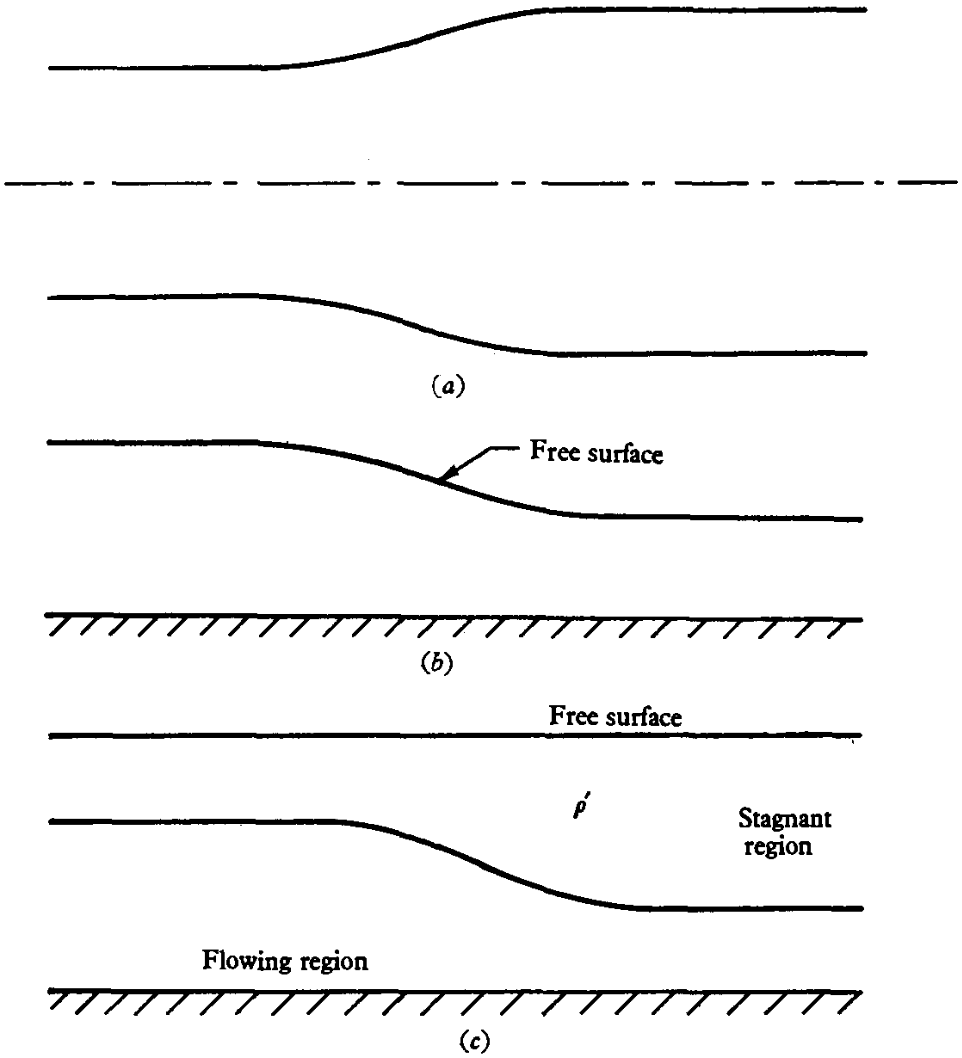


FIGURE 31. (a) The plan view of a channel contraction. (b) The elevation view of the cross section along the center plane. The fluid may be homogeneous or stratified. (c) The elevation view of the same cross section, with an overlying stagnant layer present. The flowing fluid may be homogeneous or stratified. (*Courtesy of J. Fluid Mech. and the Cambridge Univ. Press.*)

related to $\lambda(\rho)$ but has to be determined anew. To this end we need to write the Bernoulli equation for the stratified fluid, which is

$$u^2 + v^2 + 2g\left[z + \frac{1}{\rho}\int_z^h \rho(z')\,dz' - \frac{\rho'}{\rho}h\right] = C(\rho), \tag{239}$$

if h is again the depth of the flowing layer, ρ is the density in a constant-density surface, and from the upstream conditions,

$$C(\rho) = 2gh_0 B'(\rho),$$

in which

$$B'(\rho) = \left[\frac{z_0}{h_0} + \int_{z_0}^{h_0} \frac{\rho_0}{\rho_0(z_0)}\frac{dz_0'}{h_0} - \frac{\rho'}{\rho_0(z_0)}\right]. \tag{240}$$

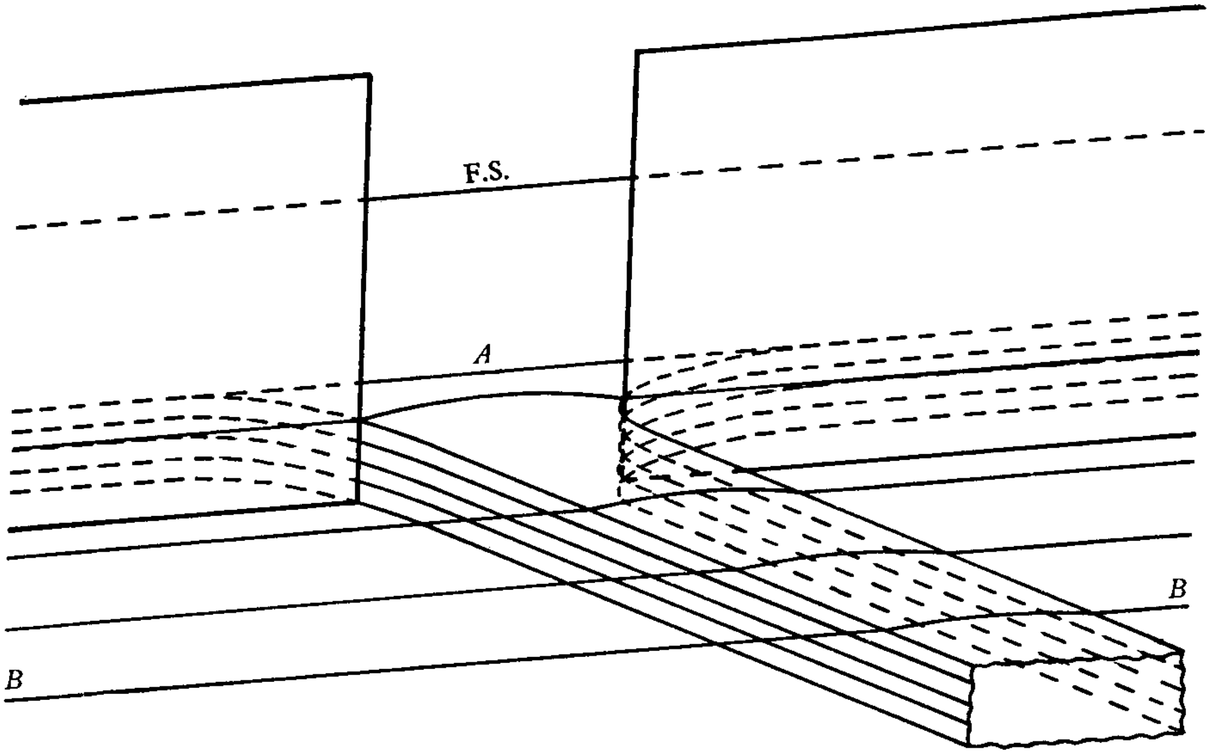


FIGURE 32. A perspective view of the gravity jet with an overlying stagnant layer. Line A again marks the elevation of the flowing fluid far upstream. The fluid may be homogeneous or stratified. F. S. is the free surface. (*Courtesy of J. Fluid Mech. and the Cambridge Univ. Press.*)

Note that $\rho_0(z_0) = \rho(z)$. Again it can be shown that the assumption is dynamically permissible and that with it the Bernoulli equation becomes

$$u^2 + v^2 + 2ghB'(\rho) = 2gh_0 B'(\rho), \tag{241}$$

or, with ρ_t denoting now the density at the top stream surface of the flowing liquid,

$$u^2 + v^2 + 2g'h \frac{\rho_t}{\rho_t - \rho'} B'(\rho) = 2g'h_0 \frac{\rho_t}{\rho_t - \rho'} B'(\rho). \tag{242}$$

Thus

$$[\lambda'(\rho)]^2 = \frac{\rho_t}{\rho_t - \rho'} B'(\rho). \tag{243}$$

Since $B'(\rho)$ is not a constant multiple of $B(\rho)$, which is given by (228) and (226), $\lambda'(\rho)$ is not proportional to $\lambda(\rho)$, given by (229). With $\lambda'(\rho)$ given by (243), (238) gives u and v, with

$$(U', V') = \frac{\rho_t - \rho'}{\rho_t} (U, V). \tag{244}$$

Of course we do not have to use the flow (U', V') parametrically and could have related u and v to U and V by inspection of (241). We have used the flow (U', V') chiefly to show more clearly that it is dangerous to apply (229) indiscriminately. Such an indiscriminate application would have given the wrong results:

$$u = \lambda(\rho)U', \qquad v = \lambda(\rho)v',$$

with $\lambda(\rho)$ given by (229). Again, it is tacitly assumed that the density far upstream in the flowing layer is obtainable from that in the stagnant liquid surrounding the jet by a stretching of the vertical length scale. This situation can be achieved as explained under Example 1, although the overlying fluid must be made level throughout by addition to the lower basin.

Example 4. Flow through a channel expansion, with an overlying stagnant layer. Figures 31a and c show the plan and elevation (through the center plane) views of a homogeneous liquid of density ρ_t flowing through a channel. The density of the stagnant layer is again ρ'. The flow is supposed to be supercritical in the sense that the speed q is everywhere greater than $(g'h)^{1/2}$, with g' given by (237). Again the velocity distribution is given by (238) and (243).

Example 5. Flow from a reservoir into a channel. Yih [1958] showed that, if a stratified liquid flows horizontally from an infinitely large reservoir into an adjoining channel with the same horizontal bottom and the same horizontal cover, the velocity distribution in the channel, where the velocity becomes unidirectional, is given by

$$\sqrt{\rho}\,u = \text{constant}. \tag{245}$$

For convenience of reference, we shall call (245) solution A, which is an inertial solution. In other words, it is true only if the acceleration is achieved by very low pressure downstream and gravity plays no role. That is to say, if we define the local Froude number as

$$V_0/\sqrt{g'h_0},$$

in which

$$g' = \frac{gh_0}{\rho}\frac{\partial p}{\partial z},$$

then the higher the minimum of the Froude number, the more nearly is the velocity distribution given by (245). At low Froude numbers, (245), though dynamically possible, is not likely to describe what actually happens, since the flow is then strongly affected by gravity. With a free surface, acceleration is caused by descent of the fluid, and the velocity distribution in a channel joining a reservoir from which the fluid issues is determined by (229), provided the downstream conditions allow such a flow. There is no contradiction of the two results. It matters a great deal whether there is a free surface, and when there is no free surface it matters a great deal how high the Froude number is.

If the upper surface is free, the velocity distribution of a homogeneous fluid issuing subcritically from the reservoir into the channel will be free from long waves, since no real characteristics exist for (221), according to the shallow-water theory. (We note, however, that shorter waves may be produced by the contraction. These cannot be studied by the shallow-water theory.) The velocity distribution far downstream from the contraction will then be uniform. That is, U will be constant according to the shallow-water theory. For a stratified fluid, downstream conditions allowing, the asymptotic velocity distribution is given by (229). For convenience we shall call this velocity distribution solution B. We know also that it is possible to have a flowing layer of a homogeneous liquid under a stagnant layer of lighter density. If we use (238) and (243) to determine u, with U' constant far downstream, we obtain a flow of a stratified fluid, with an upper part of it stagnant from the reservoir into the channel. Since the upper layer is stagnant, it indeed does not matter whether the upper surface is covered or free. This solution, called solution C, is different from solutions A and B, even granted the same upstream density distribution for the flowing layer. But solution C is valid only if blocking has occurred due to some obstacle downstream. The theorem is still true for the flowing region.

In concluding this section, we remark that there is a weakness in the gravity-jet examples. For the Kirchhoff jets the radius of curvature of the free streamline at the starting point (as it leaves the wall) is zero. The same is also true for the case of $\gamma = 2$. Sir James Lighthill suggested to the writer in London that the sharp corner at the corresponding point in the hodograph plane guarantees that the curvature at the point in question is infinite. And this turns out to be generally true. Whereas an infinite curvature is no weakness in the Kirchhoff and Chaplygin jets, it is a weakness for the gravity jets discussed here, for at a point with infinite curvature the shallow-water assumption is violated.

In all examples the flow is supposed to have originated from a large reservoir, where the fluid is at rest.

The flows discussed so far belong to a special class. Because of their very special nature it is possible to go far in the description of their detailed features. But the downstream conditions must be consistent with any particular flow belonging to this special class before it can be expected to occur, as we have described, for instance, in Example 1. It is certainly desirable to discuss the situation that will prevail for a given density distribution in the upstream reservoir and one in the downstream reservoir connected with the upstream reservoir by an open channel. This will serve to show that other flows than the class just discussed can occur.

If the free surface in the downstream reservoir is sufficiently lower than the free surface upstream, and the velocity determined by (229) is so fast that no internal waves, even of finite amplitude, can travel upstream, then the flows described by (229) will actually happen. This is true even if there are obstacles in the channel, before the obstacles are reached. The fastest speed of internal waves is of the order of the square root of the density gradient if the density gradient is continuous, or of the square root of the density difference $\Delta\rho$ divided by the mean density ρ_m if there is a density discontinuity. If the density gradient and $\Delta\rho/\rho_m$ (if an interface is present) are all very small and the velocity (U, V) determined from the free-surface drop is not small, then the solution (229) is valid. Obviously, if the free-surface drop is very small, internal waves can travel upstream, and the downstream stratification will have a far-reaching influence on the flow, which then cannot in general be described by (229), or by (238) and (244).

18. EDGE WAVES OF FINITE AMPLITUDE IN A STRATIFIED FLUID AND OTHER RESULTS

Gerstner [1802; see Lamb, 1945, p. 421] gave a particular solution for rotational waves of finite amplitude in a semi-infinite liquid of zero viscosity and constant density. It is well known that, in a frame of reference moving with Gerstner waves, the pressure on any streamline is a constant, and therefore any streamline can be considered as the trace of the free surface. This fact immediately suggests that Gerstner waves are possible in an inviscid liquid of any stratification in density, provided the depth is infinite as in Gerstner's case, and indeed the correctness of this conclusion has been fully established by Dubreil-Jacotin [1932, p. 819]. We shall now briefly present Gerstner's solution for waves (rotational) in a homogeneous fluid, and then give Dubreil-Jacotin's extension of Gerstner's solution for stratified fluids, before the edge-wave solution [Yih, 1966] is presented.

In terms of Lagrangian coordinates a_i $(i = 1, 2, 3)$, which are not necessarily the initial Cartesian coordinates, the equations of motion are

$$(\ddot{x}_\alpha - X_\alpha)\frac{\partial x_\alpha}{\partial a_i} + \frac{1}{\rho}\frac{\partial p}{\partial a_i} = 0 \qquad (i = 1,\ 2,\ 3), \tag{246}$$

in which dots indicate differentiations with respect to the time t, x_i is the ith Cartesian coordinate of a fluid particle as it moves about, X_i is the ith component of the body force per unit mass, ρ is the density, and p is the pressure. The repeated indices α imply summation over 1, 2, and 3.

Gerstner considered a two-dimensional flow independent of a_3, and took the direction of increasing x_2 to be vertical, so that

$$X_1 = X_3 = 0, \qquad X_2 = -g,$$

where g is the gravitational acceleration. Denoting x_1 and x_2 by x and y, and a_1 and a_2 by a and b, respectively, Gerstner showed that, for constant density, the solution consisting of

$$x = a + \frac{1}{k}e^{kb}\sin k(a - ct), \qquad y = b - \frac{1}{k}e^{kb}\cos k(a - ct) \tag{247}$$

satisfies (246) and the Lagrangian equation of continuity, and represents waves of finite amplitude propagating in the x-direction with a speed c given by

$$c^2 = g/k, \tag{248}$$

in which k is a wave number. The pressure is given by

$$P = p/\rho = C_0 - gb + \frac{1}{2}c^2 e^{2kb}, \tag{249}$$

in which C_0 is a constant.

For clarify in exposition we shall consider Gerstner's solution for a homogeneous liquid to be a solution satisfying

$$(\ddot{x}_\alpha - X_\alpha)\frac{\partial x_\alpha}{\partial a_i} + \frac{\partial}{\partial a_i}P = 0 \qquad (i = 1, 2), \tag{250}$$

with α ranging over 1 and 2. Now, if ρ is variable, the corresponding two equations in (246) are satisfied if

$$\frac{1}{\rho}\frac{\partial p}{\partial a_i} = \frac{\partial P}{\partial a_i}, \tag{251}$$

or

$$p = f(P), \qquad \rho = f'(P), \tag{252}$$

with the prime indicating differentiation with respect to the argument of the arbitrary function f. Since P is a function of b only, this means that the velocity field obtained by Gerstner is dynamically possible even if ρ is not constant, but is a function of b, and that the isopycnic surfaces are isobaric surfaces in Gerstner's flow. The density stratification is entirely arbitrary, and any constant-density surface can be taken to be a free surface.

Equations (251) permit one to write

$$p = \int \rho\, dP. \tag{253}$$

With P given in (249), this becomes

$$p = g\int_0^b \rho(e^{2kb} - 1)\, db = I(b), \tag{254}$$

in which the lower limit of the integral has been closen so that $p = 0$ for $b = 0$.

We now give Yih's extension [1966] of Dubreil-Jacotin's solution to produce a solution for edge waves of finite amplitude. If a coordinate system as shown in Fig. 33 is adopted, in which the direction of x (or x_1) is normal to

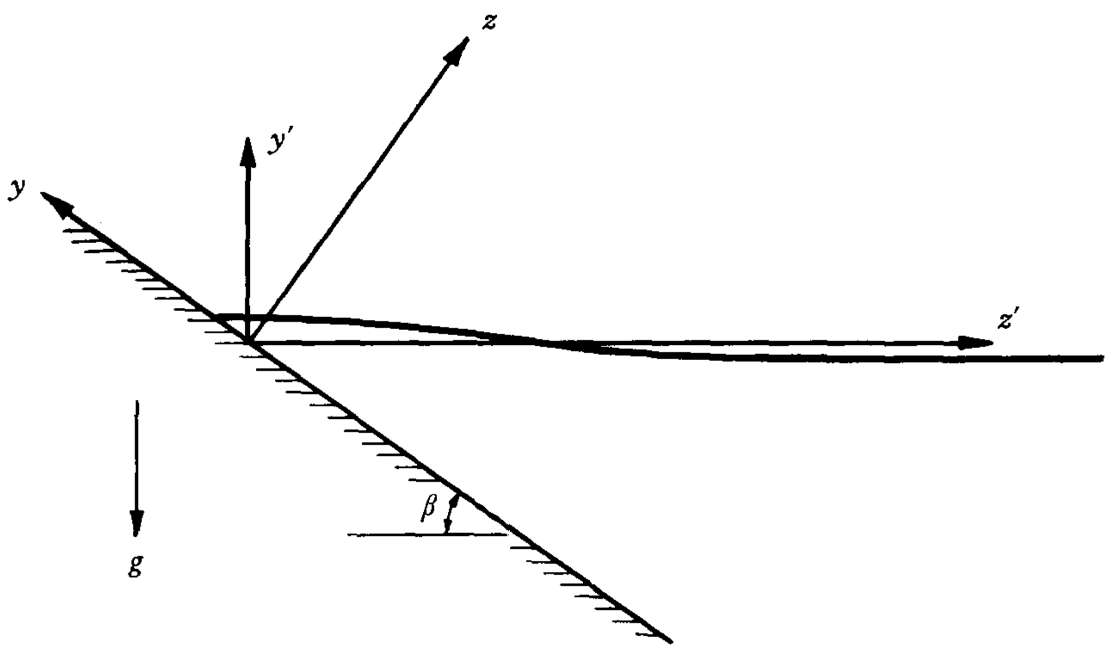


FIGURE 33. Sketch of a cross section of edge waves of finite amplitude. (*Courtesy of J. Fluid Mech. and the Cambridge Univ. Press.*)

the plane of the figure, and x_2 and x_3 are designated y and z, respectively, the body-force components are

$$X_1 = 0, \qquad X_2 = -g \sin \beta = -g_2, \qquad X_3 = -g \cos \beta = -g_3.$$

If Gerstner's velocity field is retained, the two equations in (246) for $i = 1$ and 2 are satisfied if

$$\frac{1}{\rho} \frac{\partial p}{\partial a_i} = \frac{\partial}{\partial a_i} (P - g_3 z), \tag{255}$$

with $z = a_3$, and g replaced by g_2 in the expressions for c and P in (248) and (249). The third equation in (246) merely states that p is hydrostatic in the direction of z. But (255) implies that

$$p = f(P - g_3 z), \qquad \rho = f'(P - g_3 z). \tag{256}$$

Thus again the isopycnic surfaces are isobaric surfaces, any of which can be taken to be the free surface. The density stratification is again arbitrary. With ρ or p fixed, $P \to \infty$ as $z \to \infty$. According to (247) and (249), with g in (249) identified with the present g_2, we have, as $y \to -\infty$,

$$P \sim -g_2 y + \text{constant},$$

so that $P \to \infty$ as $y \to -\infty$. Thus, from (256), the surfaces of constant ρ or p are asymptotically (for $z' \to \infty$ for fixed y') normal to the body force $\mathbf{g} = (0, -g_2 - g_3)$, as is to be expected from the vanishing of the acceleration under this limit, that is, as $z' \to \infty$ in Fig. 33.

That there is no velocity normal to the sloping shore (inclined at an angle β with the horizontal) is obvious, because Gerstner's velocity field is planar, without any component in the direction of z. It remains to mention that, since g_2 now corresponds to the g in Gerstner's original solution, it follows without further ado that the phase velocity of edge waves is given by

$$c^2 = \frac{g \sin \beta}{k}. \tag{257}$$

18.1. *Other Isolated Solutions*

One curious solution of the Euler equations for the steady flows of an inviscid and nondiffusive fluid, i.e., the first and third equations in (36) of Chapter 1, is [Yih, 1973]

$$u = U, \qquad w = -\frac{gx}{U},$$

where U is an arbitrary velocity. The streamlines (ψ is the stream function) are the parabolas

$$\psi = Uz + \frac{g}{2U} x^2.$$

It can be readily verified that the pressure p is everywhere constant, so that every streamline is a line of constant pressure and qualifies as a free streamline. The stratification, or the variation of the density ρ with ψ, is entirely arbitrary.

Solutions for circular vortex pairs and spherical vortex rings are given by Yih [1975b], who also gives the equations governing stratified and electrically conductive fluids, with or without swirl. Viscous and diffusive effects are neglected, but the flows governed by these equations are of finite amplitude.

19. FLOWS OF HOMOGENEOUS LAYERS IN A GRAVITATIONAL FIELD

The simplest case of the flow of homogeneous-fluid layers is the flow of a single layer of homogeneous fluid with a free surface. With viscous effects neglected, the flow is irrotational if it started from rest, or if, in steady flows, it originates from a section where the flow is irrotational. For irrotational flows a potential ϕ exists, the gradient of which is the velocity. In two-dimensional flows,

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \tag{258}$$

in which x and y are Cartesian coordinates, with y measured in the direction of the vertical, and u and v are velocity components in the directions of increasing x and y, respectively. The equation of continuity for two-dimensional flows of an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

This allows the use of a stream function ψ, in terms of which

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{259}$$

Equations (118) and (119) can be combined to form the Cauchy-Riemann equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x},$$

which assures that the complex potential

$$w = \phi + i\psi$$

is analytic in the complex variable $z = x + iy$. (Throughout this section, z is not used to denote the third coordinate and w does not denote the third velocity component.) Any analytic function w of z represents an irrotational flow. Since

$$\frac{dw}{dz} = u - iv,$$

the square of the speed is

$$\left|\frac{dw}{dz}\right|^2 = u^2 + v^2.$$

One well known solution (for two-dimensional flow) is of the form

$$w = Ae^{\pi i/4}z^{3/2}, \tag{260}$$

in which A is real. In polar coordinates, (260) becomes

$$w = Ar^{3/2}\left[\cos\left(\frac{3\theta}{2} + \frac{\pi}{4}\right) + i \sin\left(\frac{3\theta}{2} + \frac{\pi}{4}\right)\right].$$

Thus, the half lines

$$\theta = -\frac{\pi}{6} \quad \text{and} \quad \theta = -\frac{5\pi}{6}$$

are streamlines, which jointly constitute the free surface. The dynamical condition is

$$2gy + \left|\frac{dw}{dz}\right|^2 = \text{constant}, \tag{261}$$

and is satisfied on the free surface if

$$A = \tfrac{2}{3}\sqrt{g}.$$

Therefore, with the value of A just determined, (258) provides the solution. The streamlines (other than the free streamline) have straight asymptotes. One such streamline below and the free streamline above form a hydraulic gable.

Richardson [1920] gave two formulas for constructing free-surface flows once the free surface is given. Later writers [Lewy, 1951; Tong, 1954], who apparently were not aware of Richardson's work, independently gave similar formulas. All these formulas can be reduced to the single one [see Yih, 1957]:

$$-\frac{dz}{dw} = (2g)^{-1/3}\left\{\left[\frac{1}{H(w)} - H'^2(w)\right]^{1/2} + i\,H'(w)\right\}. \tag{262}$$

In this formula, $H(w)$ is an analytic function of w and is unrelated to the Bernoulli quantity H in previous sections, and both $H'(w)$ and the square bracket are assumed to be real on the free surface (on which dw is real, since ψ is constant). It can be easily verified that (261) is indeed satisfied.

It was Craya [1949] who used Richardson's inverse method to solve a problem of potential flow of a homogeneous fluid in contact with a stagnant fluid which is also homogeneous but of a different density. If the density of the fluid is ρ and that of the heavier one $\rho + \Delta\rho$, and if the lighter one is stagnant, the hydrostatic condition for the upper fluid is

$$p + g\rho y = \text{constant},$$

and the Bernoulli equation for the heavier fluid is

$$\frac{\rho + \Delta\rho}{2}\left|\frac{dw}{dz}\right|^2 + p + (\rho + \Delta\rho)gy = \text{constant}.$$

From these two equations it follows that

$$\left|\frac{dw}{dz}\right|^2 + \frac{2\Delta\rho}{\rho + \Delta\rho}\,gy = \text{constant}. \tag{263}$$

If the lower (heavier) fluid is stagnant, the equation corresponding to (263) is

$$\left|\frac{dw}{dz}\right|^2 - \frac{2\Delta\rho}{\rho}\,gy = \text{constant}. \tag{264}$$

If

$$g' = \frac{\Delta\rho}{\rho + \Delta\rho}\,g \qquad \text{or} \qquad -\frac{\Delta\rho}{\rho}\,g, \tag{265}$$

depending on which fluid is moving, the flow constructed from (262) by using g' instead of g will satisfy the interfacial condition (263) or (264), as can be easily verified.

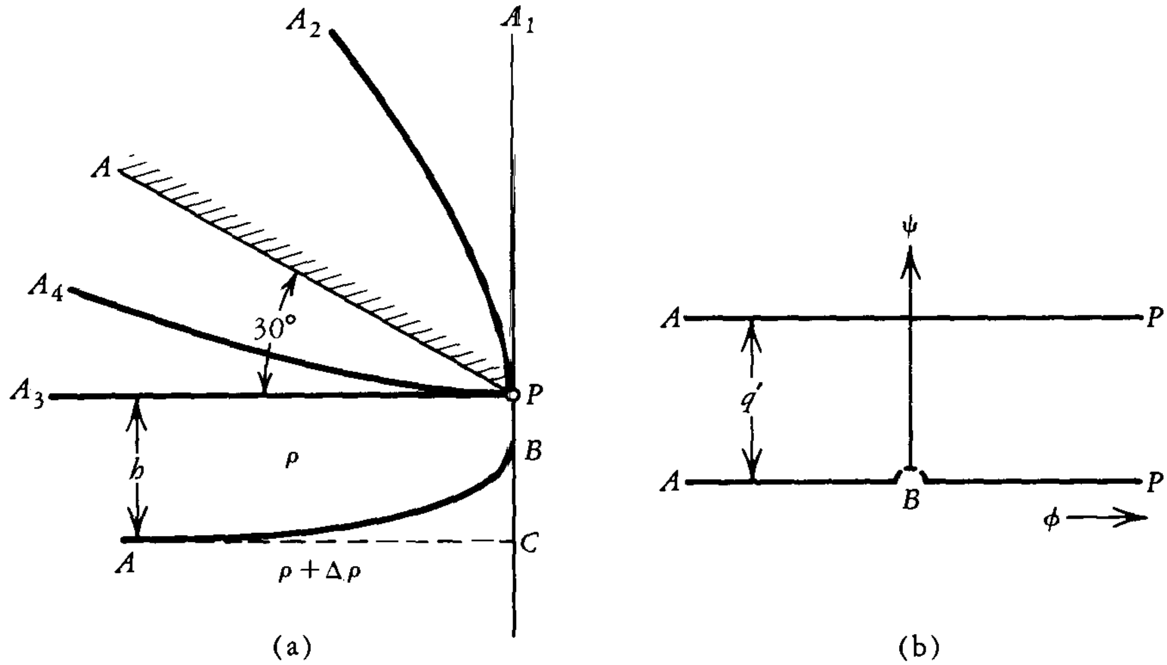


FIGURE 34. (a) Physical plane. (b) Complex-potential plane.

Craya [1949] attacked the problem of two-dimensional flow of a fluid into a sink located *above* the interface, and attempted to find the critical discharge below which the lower fluid is stagnant and above which the lower fluid can be expected to flow into the sink. Although he did not solve the problem exactly, he did find an exact solution for a similar problem, with modified upper boundary. From this solution an approximate solution for the original problem is obtained by a plausible argument. Since the flow is symmetric, it is sufficient to consider only one half of it.

In Fig. 34a, the location of the sink is indicated by P. For the original problem A_1P should be a streamline. Craya's solution corresponds to a flow with AP as a streamline. He postulated that if the flow is extended beyond AP, the streamline that enters the sink vertically will be something like A_2P, and that the difference in position between A_2P and A_1P will not change the flow below very much. Thus he was able to arrive at an approximate solution to the original problem from the exact one for which AP is a streamline.

Craya used the second expression in (265) for g' instead of g in one of Richardson's original formulas:

$$-\frac{dz}{dw} = [3g'\,G(w)]^{-1/3}\{[1 - G'^2(w)]^{1/2} + i\,G'(w)\}, \qquad (266)$$

which can be obtained from (262) by the substitution

$$H(w) = [\tfrac{3}{2}\,G(w)]^{2/3}.$$

The crucial step taken by Craya is to put

$$G = \frac{2q'}{\pi}\,e^{(\pi/2q')w}, \qquad (267)$$

in which q' is the discharge between BP and AP into the sink. Craya adopted the convention that ϕ increases in the direction of flow, so that the flow region in the w-plane is as shown in Fig. 34b.

On ABP, w and G are real. For G' less than 1, dx is not zero, but for G' greater than 1 it is. Thus B corresponds to $G' = 1$, and divides the free streamline AB from the vertical streamline BP. From A to P, $w = \phi + iq'$, G and G' are purely imaginary, and

$$\arg dz = \arg (G)^{-1/3} = -\frac{\pi}{6}.$$

Thus AP is straight and inclined to the horizontal at 30° as shown in Fig. 34a. The general features are now clear, and the next step is to find h in terms of other quantities ($\rho, \Delta\rho, g, q'$) given.

From A to B, w varies from $-\infty$ to 0. Therefore

$$CB \equiv l = (-3g')^{-1/3}\tfrac{3}{2}[G(w)]^{2/3}\Big|_{-\infty}^{0} = \left(\frac{9}{2\pi^2} F'^2 h^3\right)^{1/3}, \tag{268}$$

in which

$$F'^2 = \frac{q'^2}{g(\Delta\rho/\rho)h^3}. \tag{269}$$

From B to P, integration of (266) yields, after some straightforward simplifications,

$$i\frac{BP}{l}\int_1^t \left(\sqrt{(1/t^3) - 1} + i\right) dt, \tag{270}$$

in which

$$t = e^{(\pi/3q')w}.$$

Since $t > 1$ for $w > 0$, the radical in (270) is imaginary. As shown in Fig. 34b, the path is the dotted small circle as B is passed. Therefore (270) becomes

$$\frac{BP}{l} = \int_1^t (1 - \sqrt{1 - (1/t^3)})\, dt = 0.293.$$

Since $CB + BP = h = 1.293l$, it follows that

$$\left(\frac{9}{2\pi^2} F'^2\right)^{1/3} = \frac{1}{1.293},$$

or

$$F' = \left(\frac{1}{1.293}\right)^{3/2} \frac{\sqrt{2}\,\pi}{3} = 1.01. \tag{271}$$

Thus, if A_1P is the boundary, the *approximate* solution is

$$F = \tfrac{3}{2}F' = 1.52, \tag{272}$$

and if A_3P is the boundary, the *approximate* solution is

$$F = \tfrac{3}{4}F' = 0.75. \qquad (273)$$

Comparing (273) with Yih's solution for a continuously stratified fluid ($F = 1/\pi = 0.318$), we see that the two F's are of the same order of magnitude, so that the (critical) F for discontinuous stratification is larger, as is to be expected. Craya's calculation has been well verified by the experiment of Gariél [1949].

Aside from Craya's solution, many other solutions have been obtained with Richardson's formula, of which a number were given by Richardson himself. Unfortunately, few of these correspond to natural or realistic boundary conditions. We shall mention one for the sake of comparison with a corresponding result in Chapter 6, Section 8. Substitution of

$$H = (2g)^{1/3}H_1$$

in (262) and subsequent dropping of the subscript 1 produce

$$-\frac{dz}{dw} = \left[\frac{1}{2g\ H(w)} - H'^2(w)\right]^{1/2} + i\ H'(w), \qquad (262a)$$

which differs inconsequentially from the Richardson formula (Eq. (50) in Yih [1960e]) only because the complex velocity is now defined as the gradient of w, not the negative of it. If $H(w) = w/2V$, (262a) becomes

$$-\frac{dz}{dw} = \frac{1}{2V}\left[\left(\frac{4V^3}{gw} - 1\right)^{1/2} + i\right].$$

The velocity at infinity is vertical and equal to $-V$. From $w = -\infty$ (real) to $w = 0$ on the streamline $\psi = 0$, dz/dw is purely imaginary. At $w = 0$ the streamline $\psi = 0$ separates into two branches. From $w = 0$ to $w = 4V^3/g$ the streamline $\psi = 0$ is the cavity surface, and is given by

$$2Vz = -iw \pm \frac{4V^3}{g}\left[\sqrt{w_3(w_3 + 1)} - \cos^{-1}\sqrt{w_3} + \tfrac{1}{2}\pi\right], \qquad (274)$$

in which

$$w_3 = \frac{gw}{4V^3}.$$

From $w = 4V^3/g$ to $w = \infty$ (real), dz/dw is again purely imaginary, so that the solid boundaries corresponding to this range of w and $\psi = 0$ are vertical straight lines. The cavity form is given in Fig. 35.

The problem of a gas bubble rising in a tube filled with liquid has been investigated by Garabedian [1957]. Since the flow is axisymmetric rather than two-dimensional and a free surface is involved, the solution is very difficult, particularly since gravity must be taken into account. Garabedian used a minimizing procedure, and found a class of solutions. The lack of uniqueness in

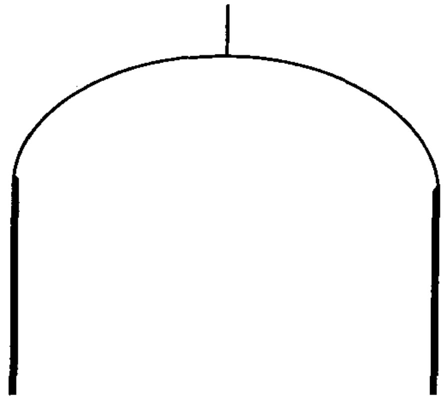


FIGURE 35. Form of cavity in a potential flow with the effect of gravity included. (*Proc. Roy. Soc.* (*A*), *245*, *pp. 312–329. Courtesy of the Royal Society of London.*)

the solution of potential-flow problems involving a free surface will again be illustrated in Section 6 of Chapter 5.

If two superposed layers are both flowing, the transformations are more complicated. These have been given by Yih [1957]. If ρ_1 and ρ_2 are the densities of the two fluids ($\rho_2 > \rho_1$), and

$$E = (2g')^{1/3}, \qquad g' = g\,\frac{\rho_2 - \rho_1}{\rho_2}, \qquad k = \frac{\rho_1}{\rho_2},$$

the transformations are

$$\begin{aligned} -\frac{dz}{dw_1} &= \frac{1}{E}\,[(kF_1^{-1} - G_1'^2)^{1/2} + iG_1'], \\ -\frac{dz}{dw_2} &= \frac{1}{E}\,\{[(F_1 + G_1)^{-1} - K_2'^2]^{1/2} + iK_2'\}, \end{aligned} \tag{275}$$

in which w_1 and w_2 are the complex potentials of the upper and lower fluids, respectively, F_1 and G_1 are functions of w_1, and K_2 a function of w_2. On the free (or interfacial) streamline, G_1 and K_2 and the two radicals in (275) are real. It can be readily verified that the eliminant of the Bernoulli equations for the two fluids along the free streamline, namely

$$\rho_1\left|\frac{dw_1}{dz}\right|^2 - \rho_2\left|\frac{dw_2}{dz}\right|^2 = 2g(\rho_2 - \rho_1)y, \tag{276}$$

is satisfied by (275) on that streamline. The several functions of w_1 and w_2 are arbitrary, except that geometrical compatibility requires that, on the free streamline,

$$G_1(w_1) = K_2(w_2), \tag{277}$$

and

$$(kF_1^{-1} - G_1'^2)^{1/2}\,dw_1 = [(F_1 + G_1)^{-1} - K_2'^2]^{1/2}\,dw_2. \tag{278}$$

For more detailed discussions, see Yih [1957].

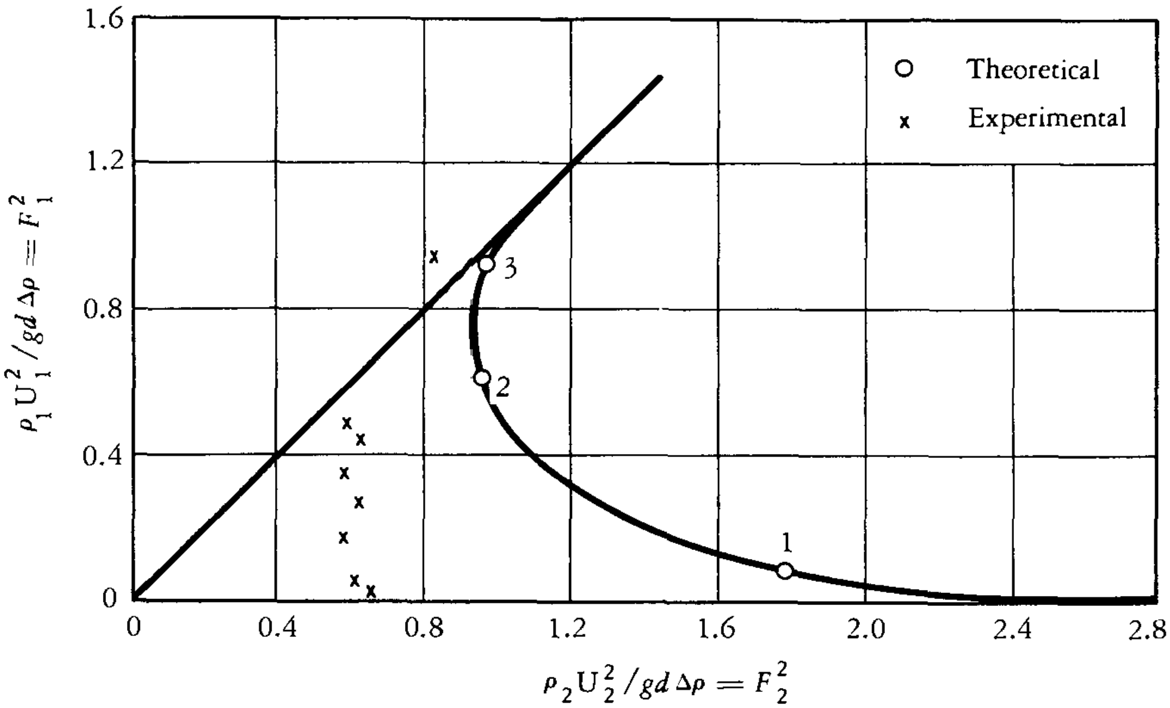


FIGURE 36. Chart for discharges into a sink from two fluid layers, after Huber [1960]. The theoretical curve has been obtained by the relaxation method, and the experimental points were obtained by using a layer of water superposed on a layer of sugar solution. The depths are equal for both theory and experiment. d = depth of each layer. ρ_1 = density in upper layer. ρ_2 = density in lower layer. $\Delta\rho = \rho_2 - \rho_1$. U_1 = velocity in upper layer far upstream. U_2 = velocity in lower layer far upstream. g = gravitational acceleration. (*Courtesy of the American Society of Civil Engineers.*)

The problem of the flow of two layers of equal depths (in a channel of total depth $2d$) into a sink located at the intersection of the bottom and a vertical wall has been considered by Huber [1959]. Using the relaxation method, he not only determined the critical Froude number, but also the ratio of discharges of the two layers as a function of the Froude number, after its critical value is exceeded. With U_1 and U_2 denoting the uniform upstream velocities of the two layers and the Froude number defined by

$$F = \frac{q}{\sqrt{g[(\rho_2 - \rho_1)/\rho_2]d}\, d},$$

the result of his calculations are shown in Fig. 36, from which it can be seen that the critical value of F^2 is 2.74, or that the critical value of F is 1.66. As $F \to \infty$, $U_1/U_2 \to \sqrt{\rho_2/\rho_1}$, on the assumption that the flow originates horizontally from a big reservoir. Huber's critical F is more than twice that of Craya (0.75). Experiments performed by Huber (unpublished) tend to show a lower critical value for F than 1.66. How much of this discrepancy is due to viscous effects is not yet known. In this connection it should be remembered that Craya's critical F is also not exact.

Harleman *et al.* [1959] have performed experiments to determine the critical Froude number for axisymmetric flow of a layer of liquid into a sink. A stagnant layer of a lighter liquid, of the same asymptotic depth (d), lies over the flowing liquid, and the critical Froude number is

$$F_{cr} \equiv \frac{Q_{cr}}{\sqrt{g'd}\, d^2} \simeq \frac{\pi}{2} = 1.57, \tag{279}$$

with

$$g' = g\,\frac{\rho_2 - \rho_1}{\rho_2},$$

and Q_{cr} denoting the critical discharge. Harleman *et al.*'s data were plotted in such a way that at first sight one gets the impression that Q_{cr} varies sensitively with the size of the opening of the sink. This impression is a wrong one, and with the Froude number defined above their data really show that Q_{cr} is essentially independent of D, so long as it is not too large. This is a most welcome situation.

20. INTERNAL HYDRAULIC JUMPS

Two or more layers of homogeneous fluids flowing horizontally may suffer an abrupt change in depths. This abrupt change in depths is called an internal hydraulic jump. Internal hydraulic jumps in two superposed fluid layers with a free surface and an interface have been studied by Yih and Guha [1955]. The analysis is based on the assumption that the pressure distribution at the interface is hydrostatic and that interfacial shear as well as shear at the bottom of the channel can be neglected. Thus, with reference to Figs. 37a and 37b, the mean head over the jump section is $(h_1 + h_1')/2$. If now q denotes the discharge, ρ the density, and g the gravitational acceleration, and if subscripts 1 and 2 are used to indicate the upper and lower layers, respectively,

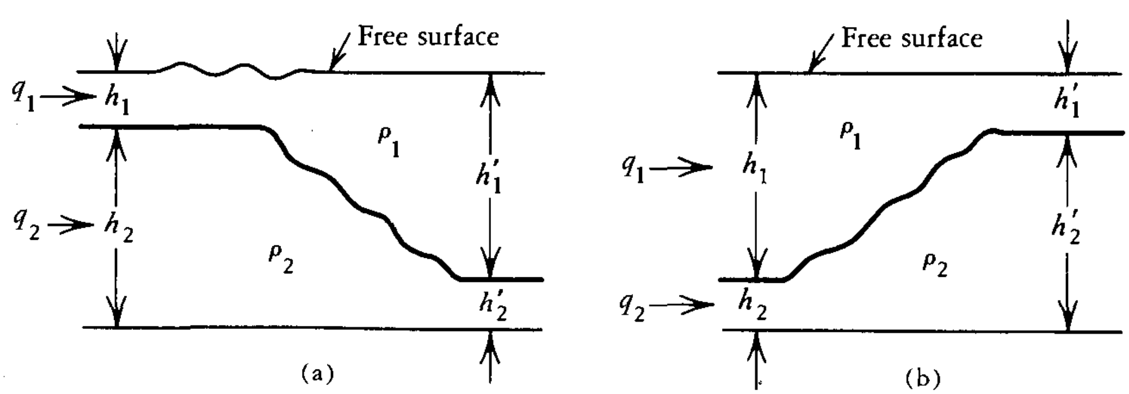


FIGURE 37. Definition sketch. (a) Normal jump. (b) Inverted jump. (*Courtesy of Tellus.*)

the application of the momentum principle to the lower layer results in the equation

$$\rho_2 q_2^2\left(\frac{1}{h_2'} - \frac{1}{h_2}\right) = h_2 h_1 \rho_1 g + \frac{1}{2} h_2^2 \rho_2 g + \frac{1}{2}(h_1 + h_1')(h_2' - h_2)\rho_1 g - h_2' h_1' \rho_1 g - \frac{1}{2} h_2'^2 \rho_2 g. \qquad (280)$$

The other quantities in (280) are defined in Figs. 37a and b. With

$$a_2 = \frac{q_2^2}{g}, \qquad r = \frac{\rho_1}{\rho_2},$$

(280) can be rewritten as

$$2a_2(h_2 - h_2') = h_2 h_2'(h_2 + h_2')[r(h_1 - h_1') + (h_2 - h_2')]. \qquad (281)$$

Similarly, for the upper layer

$$2a_1(h_1 - h_1') = h_1 h_1'(h_1 + h_1')[(h_1 - h_1') + (h_2 - h_2')], \qquad (282)$$

in which

$$a_1 = \frac{q_1^2}{g}.$$

It has been shown [see Yih and Guha, 1955] that (281) and (282) can have at most nine real solutions, one of which is obviously $h_1' = h_1$ and $h_2' = h_2$. But of these nine solutions five are not entirely positive (that is, positive in h_1' and h_2'). Hence there can be at most only four positive solutions representing four mutually conjugate states. In Fig. 38c, the four solutions are given by the coordinates of the four points 1, 2, 3, and 4, any of which may represent the given upstream state. These points are the points of intersection of a branch of (282), labeled III, and a branch of (281), labeled VI. In Figs. 38a and 38b, there are only two points of intersection. The point 1 corresponds to smaller depths, and it can be shown that the mechanical energy associated with it is greater than that associated with the point 2. Hence it is possible for the layers to jump from state 1 to state 2, but impossible for the reverse to occur. Since $h_1 < h_1'$ and $h_2 < h_2'$, the jump is not primarily an internal jump.

Internal jumps can occur only when there are four positive solutions of (281) and (282). In Fig. 38c, the four conjugate states indicated by the four points of intersection correspond in general to four different specific mechanical energies. Point 1 has the greatest specific energy. If the other points are numbered in descending order of their specific energies, a jump from a state with a larger index to one with a smaller index is impossible, but the reverse can occur. Thus, if the given state is indicated by the point 4, no jump can occur. If it is indicated by the point 3, the downstream state can only be state 4 if a jump occurs. If it is indicated by the point 2 or 1, the post-jump state is not unique, and which state will prevail downstream is determined by the controls downstream.

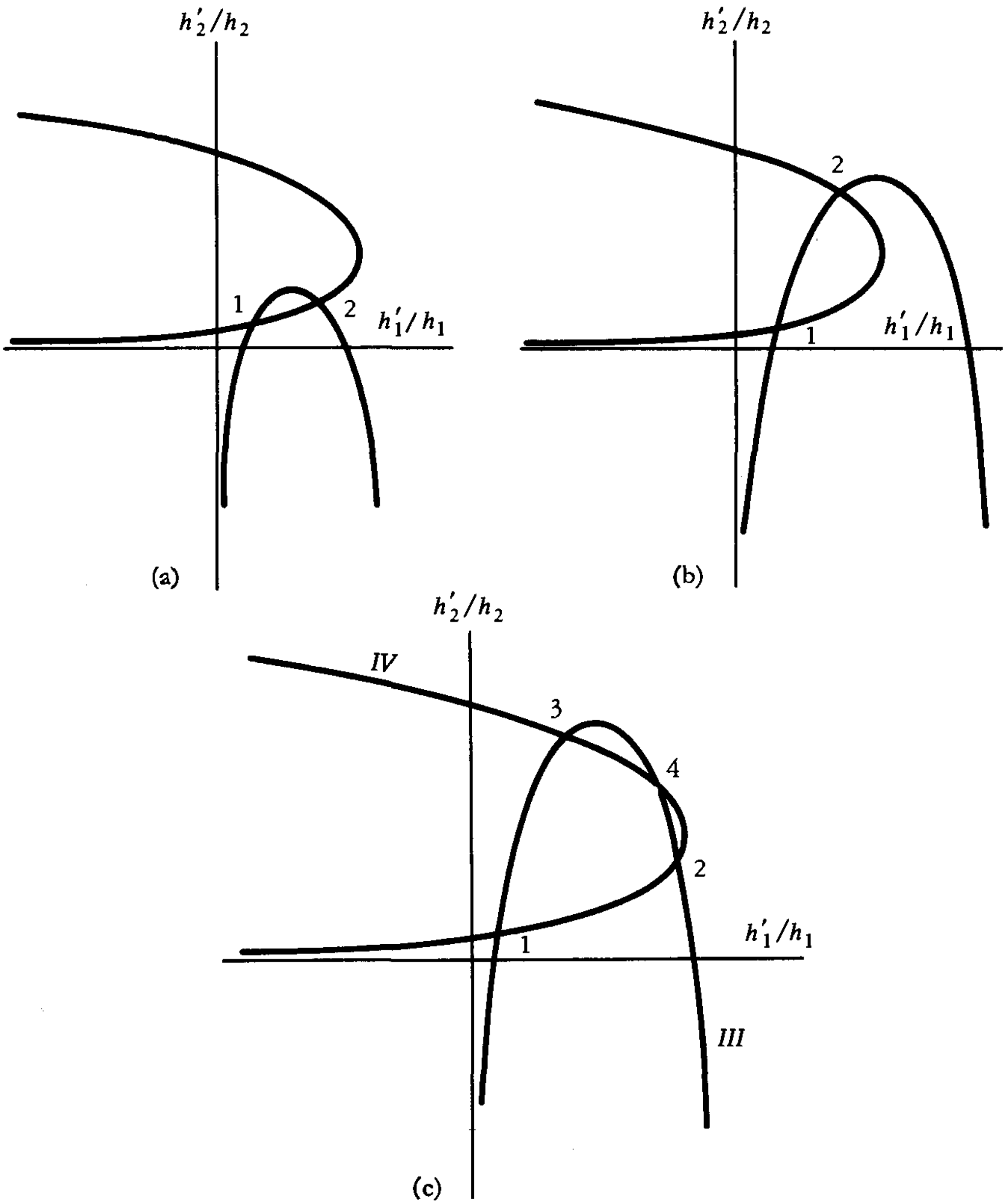


FIGURE 38. Sketch showing possible conjugate states for an internal hydraulic jump. (a) and (b) Both depths increase as the jump bridges State 1 and State 2 (downstream). (c) Four conjugate states.

An example illustrating how downstream constraints can make a jump unique is furnished by the case in which the downstream velocities for the two layers are the same. This condition can be easily realized in the laboratory by a closed tail gate in a channel, giving rise to a moving hydraulic jump. In this case

$$\frac{q_1}{q_2} = \frac{h'_1}{h'_2} = \lambda, \tag{283}$$

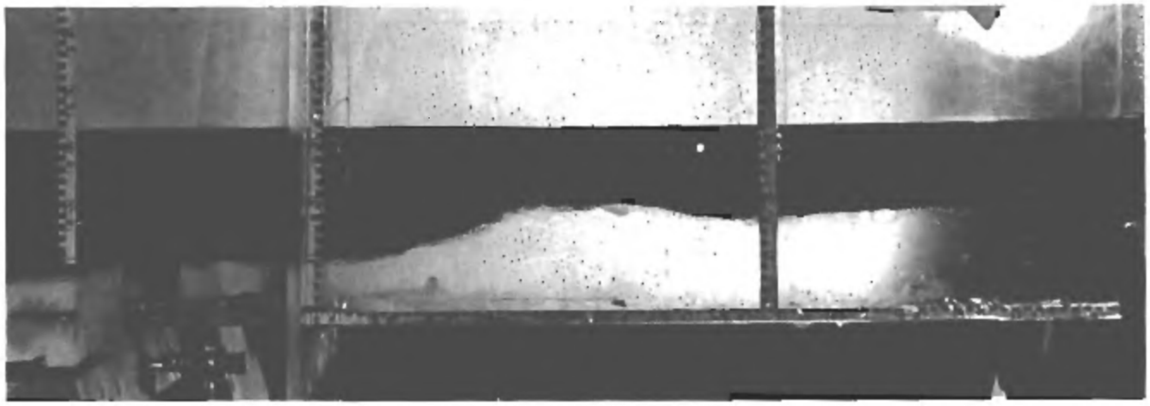


FIGURE 39. Photograph of an internal jump with the downstream velocity equal in both layers. (*Courtesy of Tellus.*)

and (281) becomes a third-degree equation in h_2', which has two positive roots. One root obviously corresponds to the given state, and the other the conjugate state. Figure 39 shows the jump obtained by Yih in a lucite flume 4 feet long, 6 inches wide, and 8 inches deep. The gate downstream was closed, and the sluice gate upstream was first closed and either stanisol (an oil prepared by the Standard Oil Company, with a specific gravity of 0.777) or a mixture of stanisol and carbon tetrachloride (with a specific gravity of 1.59) was introduced into the flume. Then the sluice gate was opened to admit water into the flume. As the water flowed into the flume, oil was allowed to flow through horizontal slots on the side of the channel near the upstream end, so that the depths were maintained constant upstream. At first the water moved downstream in a surge. Then it hit the closed tail gate and an internal hydraulic jump was formed and moved upstream. The speed of its propagation was measured by stop-watch timing and a tape attached to the flume to measure distance traveled by the jump. The free surface was observed to be nearly horizontal. The discharge of water into the flume, equal to the discharge of oil (or oil mixture) out of the flume, can be computed from the upstream and downstream depths recorded photographically and the measured speed of propagation of the jump. The actual downstream velocities of both layers were of course zero. Now all the velocities can be taken relative to the jump front, and Eqs. (281) and (283) be used to calculate h_2'. The comparison of the results of this calculation (which, to be sure, had been based *partially* on the measured h_2') with the measured h_2' is shown in dimensionless terms in Fig. 40, from which it can be concluded that the assumptions of hydrostatic pressure distribution at the interface and the negligibility of shear at the bottom and at the interface are not far from reality, and that downstream constraints can indeed make the downstream state unique. Other experiments on internal

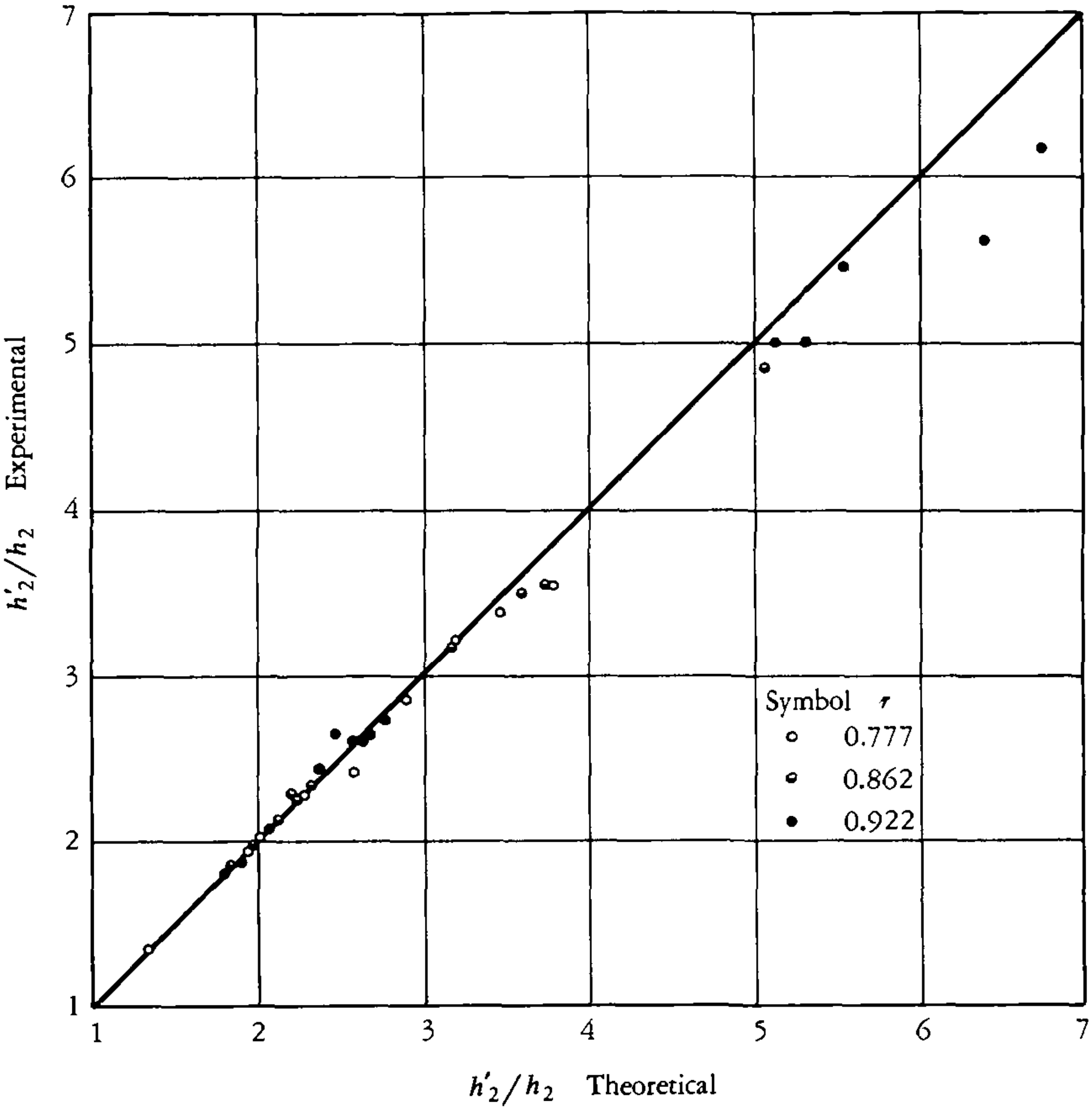


FIGURE 40. Comparison between the theory and the experimental results for the internal jump with equal downstream velocities. (*Courtesy of Tellus.*)

hydraulic jumps have been done by Guha [see Yih and Guha, 1955] and Stommel and Farmer [1952].

Hydraulic jumps in fluid systems of more than two layers can be analyzed in a similar fashion. If the density stratification is continuous, there may exist conjugate states satisfying the continuity equation and the momentum equation. But no such states are known which have different mean specific energies. Since hydraulic jumps are characterized by a loss of energy, it is questionable whether the change from one state to the other without energy loss can be called a hydraulic jump. Furthermore, should such states be known to have different mean specific energies and a proper jump be possible, the turbulent mixing causing the energy loss across the jump must be taken into account as a factor affecting the downstream condition. In fact this mixing is already present in internal hydraulic jumps of two homogeneous but miscible layers, though one can neglect it without committing so much error as to make the results meaningless. For continuous stratifications, it is probably not negligible. At the present time, hydraulic jumps in a continuously stratified fluid have not been properly analyzed or understood, or yet observed if the stratification is everywhere gradual.

21. INTERNAL SURGES

The intrusion of a heavy fluid into a lighter one is not uncommon in nature. It occurs when a cold air current advances in a warm atmosphere, when a muddy stream enters a reservoir of clear water, or when tides cause wedges of salt water to progress upstream in estuaries. In industry it occurs, for instance, in the process of manufacturing glass. When a cold current advances with sufficient speed over dusty grounds, it can pick up dust particles which in turn make the cold air mass even heavier (see frontispiece). Such currents are sometimes called density currents, especially in the literature of hydraulics. But it is well to remember that density difference alone does not cause the currents, and that it is the difference in specific weights that really causes them. To recognize the dominant role played by gravity in such currents, they will be called gravity currents in this book, after Hunter Rouse.

The speed of a gravity current advancing in an ambient fluid was considered by von Kármán [1940]. In Fig. 41a, the depth of the current is denoted

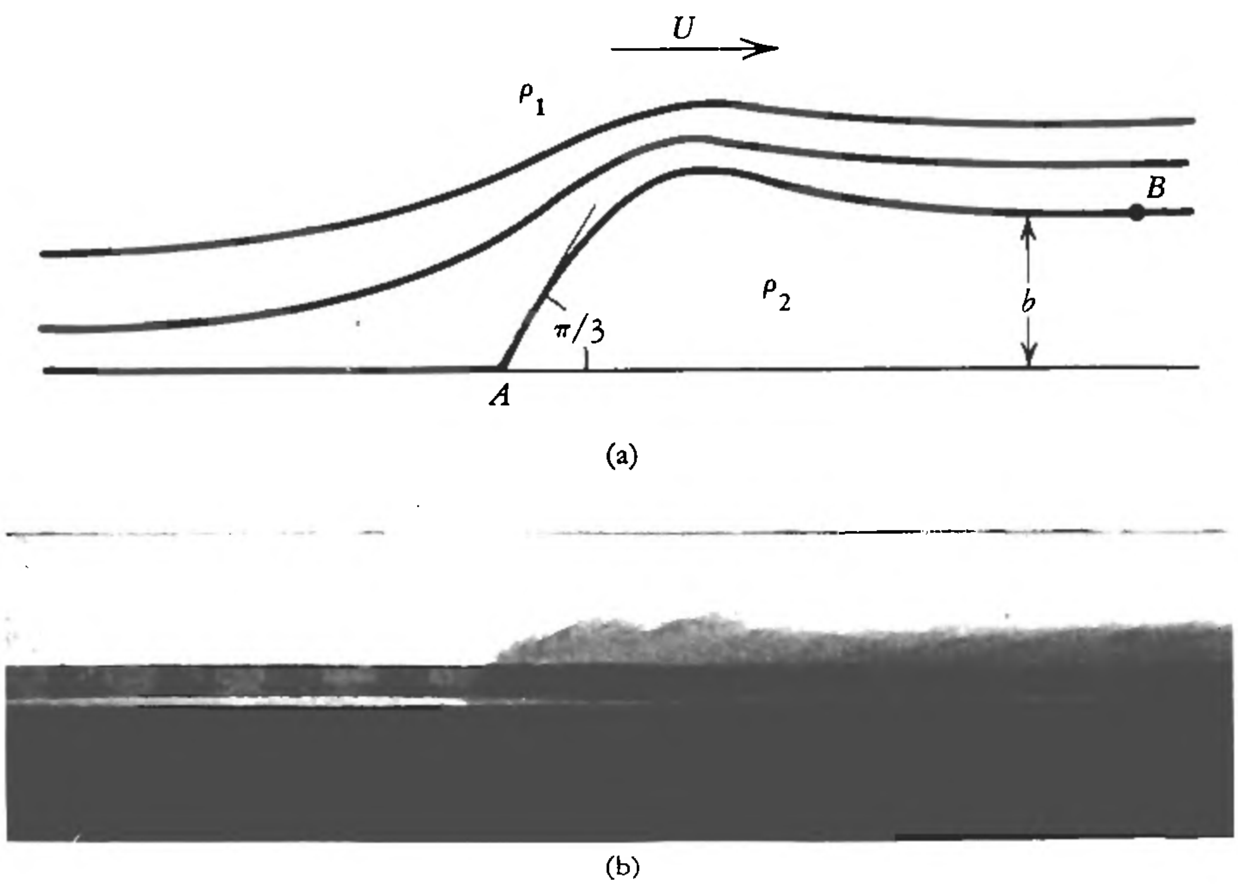


FIGURE 41. (a) Sketch of a gravity current. (b) Photograph of a gravity current.

by b. If the density of the fluid constituting the current is ρ_2, and that of the infinite ambient fluid is ρ_1 ($<\rho_2$), the speed of advance of the current, U, can be found in terms of ρ_1, ρ_2, b, and the gravitational acceleration g. The flow can first be made steady by superposing a uniform velocity U (equal and opposite to the actual velocity of the gravity current) to the entire field of flow. The current is then brought to rest. Application of the Bernoulli equation to points A and B on the boundary of the current yields

$$p_A = p_B + g\rho_1 b + \frac{\rho_1}{2} U^2.$$

But, since the current is now at rest,

$$p_A - p_B = g\rho_2 b.$$

From the last two equations it follows that

$$U = \sqrt{2bg'}, \qquad \text{with} \qquad g' = \frac{g(\rho_2 - \rho_1)}{\rho_1}. \tag{284}$$

If the lighter fluid is not infinite in extent, the result is quite different. The case of mutual intrusion (Fig. 42) of two fluids of slightly different specific weights originally of the same depth and separated by a gate in a flume was considered by Keulegan [1958—but the work was done before 1947] and Yih [1947], both of whom used fresh water and salt water for their experiments. The analysis can best be done in terms of energy. In time interval Δt, the wedges will have advanced a distance $U\,\Delta t$, since for all practical purposes the two wedges can be assumed antisymmetric, as shown in Fig. 42. Thus the lighter fluid will have gained potential energy by the amount $[g\rho_1(b/2)]\,(U\,\Delta t\, b)$ per unit width of the channel. The potential energy lost (per unit width) by the heavier fluid will be $[g\rho_2(b/2)]\,(U\,\Delta t\, b)$. The kinetic energy gained by both fluids is

$$\tfrac{1}{2}(\rho_1 + \rho_2)U^2(bU\,\Delta t)$$

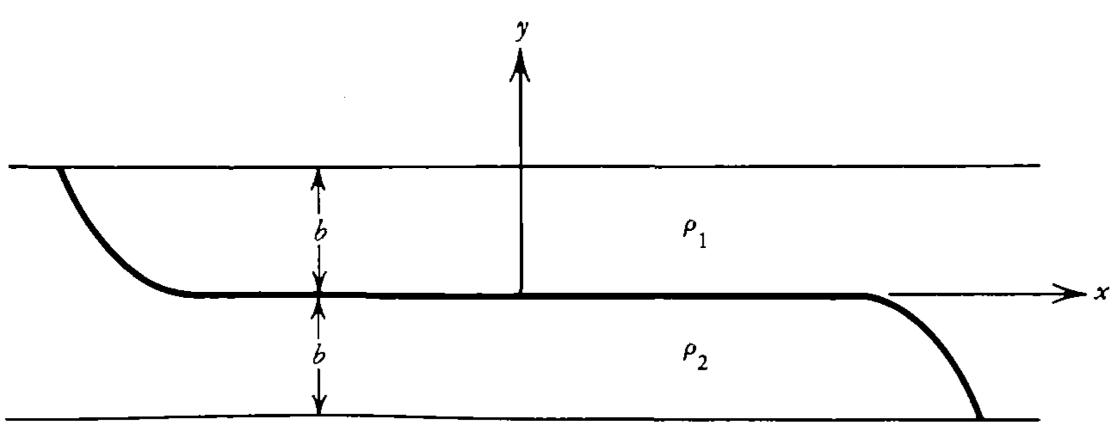
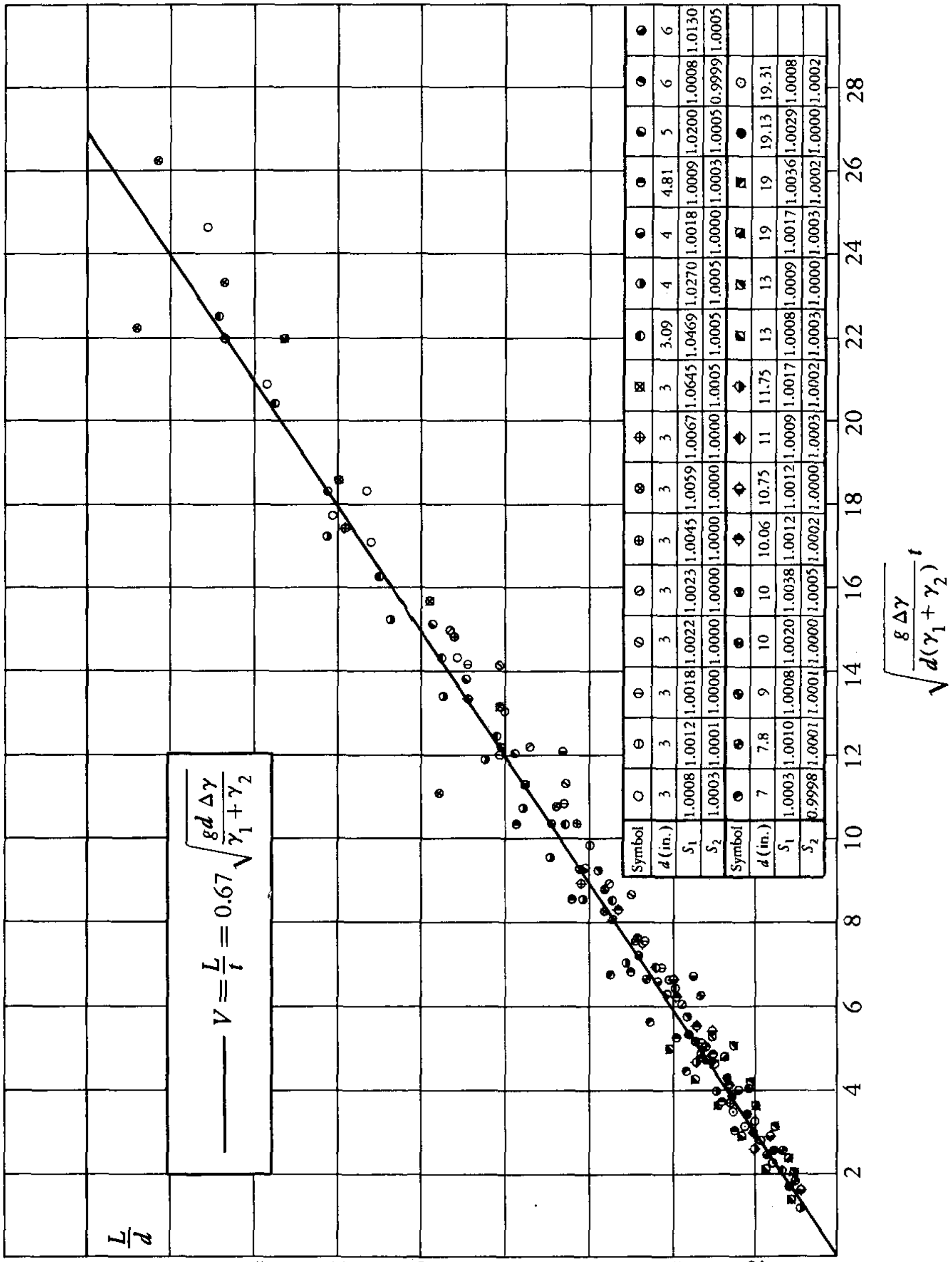


FIGURE 42. Sketch for a case of mutual intrusion.



Symbol	○	⊖	⦶	⊘	⦸	⊕	⊗	⊕	⊠	◒	◓	◐	◑	◐	◑	◐
d (in.)	3	3	3	3	3	3	3	3	3	3.09	4	4	4.81	5	6	6
S_1	1.0008	1.0012	1.0018	1.0022	1.0023	1.0045	1.0059	1.0067	1.0645	1.0469	1.0270	1.0018	1.0009	1.0200	1.0008	1.0130
S_2	1.0003	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0005	1.0005	1.0005	1.0000	1.0003	1.0005	0.9999	1.0005
Symbol	◔	⊛	⊛	⊛	⊛	◈	◈	◈	◈	◩	◩	◪	◪	●	⊙	
d (in.)	7	7.8	9	10	10	10.06	10.75	11	11.75	13	13	19	19	19.13	19.31	
S_1	1.0003	1.0010	1.0008	1.0020	1.0038	1.0012	1.0012	1.0009	1.0017	1.0008	1.0009	1.0017	1.0036	1.0029	1.0008	
S_2	0.9998	1.0001	1.0001	1.0000	1.0005	1.0002	1.0000	1.0003	1.0002	1.0003	1.0000	1.0003	1.0002	1.0000	1.0002	

per unit width. Invoking the principle of the conservation of energy, we have

$$(\rho_1 + \rho_2)U^2 = gb(\rho_2 - \rho_1),$$

or

$$U = \sqrt{\frac{gb(\rho_2 - \rho_1)}{\rho_2 + \rho_1}}. \tag{285}$$

It can be seen that (285) differs from (284) only in that ρ_1 in the denominator in (284) is replaced by $\rho_1 + \rho_2$ in (285). This means that the inertial effect of the heavier fluid is important when there is a change in its velocity (from zero to U). With $d = 2b$, (285) becomes

$$U = 0.71\sqrt{\frac{gd(\rho_2 - \rho_1)}{\rho_2 + \rho_1}}. \tag{286}$$

The experiments of Yih [1947] showed that the experimental data can be represented closely by the formula

$$U = 0.67\sqrt{\frac{gd(\rho_2 - \rho_1)}{\rho_2 + \rho_1}}. \tag{287}$$

The data are shown in Fig. 43, together with the line representing (287).

A saline wedge intruding upstream into a flowing fresh-water stream can be arrested if the velocity of that stream is sufficiently high. The interfacial mixing and significant stresses of arrested saline wedges have been studied by Keulegan [1955a, -b], Lofquist [1960], and Bata [1959].

22. UNIDIRECTIONAL LAMINAR FLOWS

Laminar flows of a stratified fluid in a long channel or conduit can be determined exactly if diffusivity is neglected. Two cases will be considered: the case of two-layer flow and the case of continuous stratification.

The equations of motion for unidirectional flow of two wide superposed homogeneous-fluid layers along a channel with angle of inclination β are

$$\mu_1 \frac{d^2u_1}{dy^2} = \frac{\partial p}{\partial x} - \rho_1 g \sin\beta = K_1, \tag{288}$$

and

$$\mu_2 \frac{d^2u_2}{dy^2} = \frac{\partial p}{\partial x} - \rho_2 g \sin\beta = K_2, \tag{289}$$

in which $\partial p/\partial x$, and therefore K_1 and K_2, are constants. The coordinates are shown in Fig. 44. The subscripts 1 and 2 are used to denote the upper and the

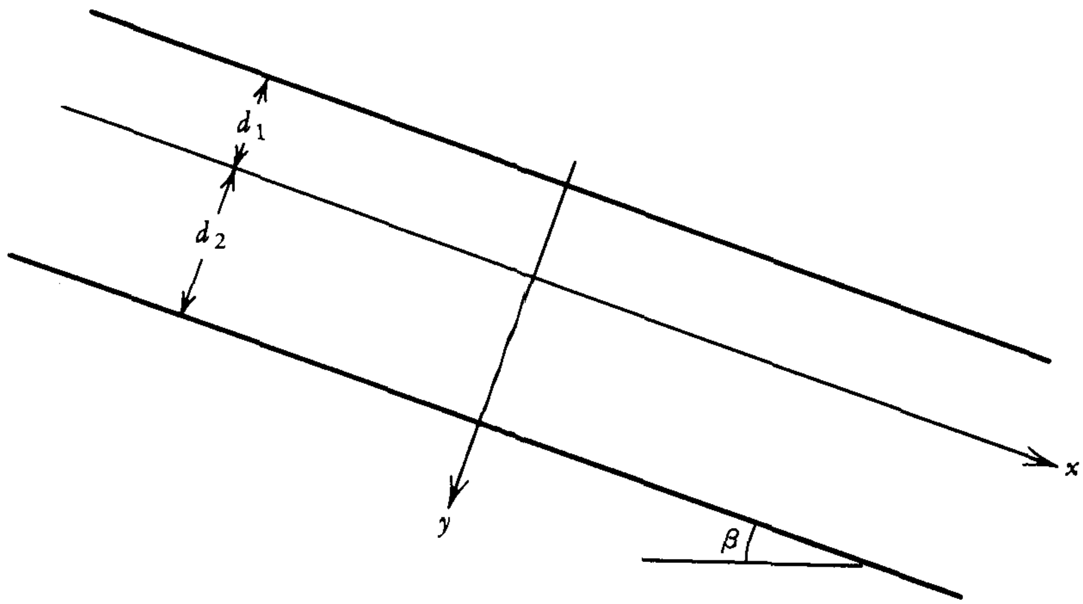


FIGURE 44. Coordinates for two-layer flow down an inclined plane.

lower layer, and μ, p, ρ, and g denote the viscosity, pressure, density, and gravitational acceleration, respectively. The solutions of (288) and (289) are

$$u_1 = \frac{K_1}{\mu_1}(y^2 + C_1 y + D_1), \qquad u_2 = \frac{K_2}{\mu 1}(y^2 + C_2 y + D_2).$$

If the upper and lower boundaries are fixed, $u_1 = 0$ at $y = -d_1$, and $u_2 = 0$ at $y = d_2$. At the interface ($y = 0$) $u_1 = u_2$, and

$$\mu_1 \frac{du_1}{dy} = \mu_2 \frac{du_2}{dy},$$

because the velocity and the shear stress must be continuous there. The four boundary conditions give

$$C_1 = \frac{md_1^2 - kd_2^2}{md_1 + d_2}, \qquad C_2 = kC_1,$$

$$D_1 = -\frac{d_1 d_2(d_1 + kd_2)}{md_1 + d_2}, \qquad D_2 = \frac{m}{k}D_1,$$

in which

$$k = \frac{K_2}{K_1} \quad \text{and} \quad m = \frac{\mu_2}{\mu_1}.$$

If the upper boundary is free, the condition $u = 0$ at $y = -d_1$ is replaced by $du_1/dy = 0$ at $y = -d_1$, and the constants are

$$C_1 = 2d_1, \qquad C_2 = kC_1,$$

$$D_1 = -\frac{d_2}{m}(2d_1 + kd_2), \qquad D_2 = \frac{m}{k}D_1.$$

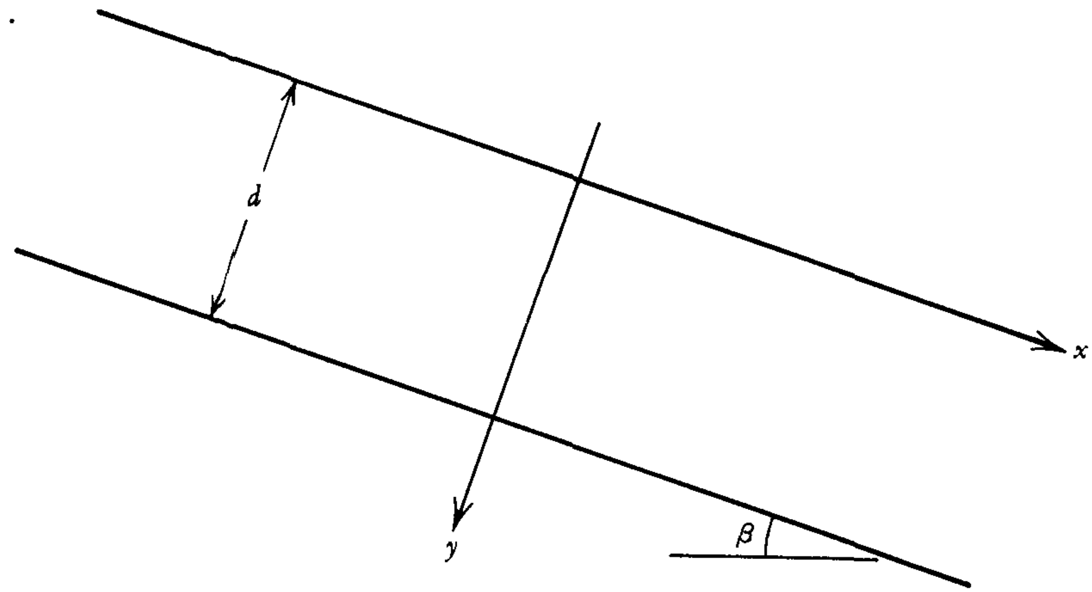


FIGURE 45. Coordinates for continuously stratified fluid down an inclined plane.

Problems involving more than two layers can be treated in a similar fashion.

The equation governing unidirectional flows of a continuously stratified fluid in a wide channel (Fig. 45) is

$$\frac{d}{dy}\left(\mu \frac{du}{dy}\right) = \frac{dp}{dx} - \rho g \sin \beta,$$

in which μ and ρ are now functions of y. Two quadratures give the result

$$u = \int_d^y \left[\frac{1}{\mu}\int_0^y \left(\frac{dp}{dx} - \rho g \sin \beta\right) dy\right] dy \tag{290}$$

if the upper boundary is free, and the result

$$u = \int_d^y \left[\frac{1}{\mu}\int_0^y \left(\frac{dp}{dx} - \rho g \sin \beta\right) dy + C\right] dy, \tag{291}$$

in which C is determined by

$$Cd = \int_d^0 \left[\frac{1}{\mu}\int_0^y \left(\frac{dp}{dx} - \rho g \sin \beta\right) dy\right] dy,$$

if the upper boundary is fixed.

If the conduit is a rectangular or circular pipe, the solution is already difficult is μ is not constant. However, if μ is constant the solution can be readily obtained by first finding, by quadratures, a particular solution $u_p(y)$ to get rid of the nonhomogeneous part of the equation. The complementary solution u_c then satisfies the Laplace equation in two variables, in either Cartesian or polar coordinates. This equation, with appropriate boundary conditions, can be solved by the method of separation of variables. The details of the solution need not be presented here.

23. GRAVITATIONAL CONVECTION FROM SOURCES

Laminar gravitational convection from a maintained source, as evidenced by the initial plume of a cigarette, has been considered by Yih [1951, 1952], who gave solutions in closed form of the simultaneous nonlinear equations governing the flow, for certain special values of the Prandtl number, and for both point and line sources. The corrected versions of these solutions are given in Yih [1953]. Results for turbulent gravitational convection from a maintained point or line source in an unstratified atmosphere are given in Schmidt [1941], Yih [1951], and Rouse *et al.* [1952]. These results have been summarized in Yih [1953]. Turbulent gravitational convection from maintained or instantaneous sources in a stratified atmosphere has been investigated by Morton, Taylor, and Turner [1956]. See also Scorer [1959] and Lee and Emmons [1961].

NOTES

Section 6

1. Koh [1966] considered very slow two-dimensional motion of a linearly stratified viscous and diffusive fluid into a line sink. He started with the boundary-layer equations, one consequence of which is the hydrostatic pressure distribution. After the main equation of motion and the full diffusion equation were made dimensionless, in terms of the dimensionless space variables ξ (proportional to the horizontal and longitudinal distance x) and η (proportional to the vertical distance y), he used a new variable $\zeta = \eta/\xi^{1/3}$. But a similarity solution is possible only for very slow motion, for which the acceleration and the nonlinear convective terms (though not the main convective term, which is linear) are neglected. The full boundary-layer equations are solvable only if expansions in powers of $\xi^{-2/3}$, with functions of ζ as coefficients, are used for the stream function and the salinity. For very slow motion the stream function is a function of ζ and the salinity perturbation is $\xi^{-1/3}$ times a function of ζ.

When Koh's Stokes-flow solutions were compared with his experiments, the agreement was excellent.

Yet the solution also raises questions. First of all, the parabolas $\zeta =$ constant extend from the origin to very large values of η (or of y), positive and negative. The density stratification being linear, for large y the density is negative, and this poses some difficulty toward infinity. This difficulty may be obviated by restricting the region of validity of the solution to a region far, but not too far, away from the sink (near which the velocity is high). Second, the limiting case of no density gradient, that is, of a homogeneous fluid, cannot be obtained from the solution given by letting the density gradient approach zero, for the pressure distribution in a genuine Stokes flow of a homogeneous fluid is not hydrostatic, since viscosity plays an important role in its determination. Quite possibly Koh's solution is valid only if the parameter

$$-\frac{g}{vq}\left(\frac{1}{\rho_0}\frac{d\rho_0}{dy}\right)$$

is large, where g is the gravitational acceleration, v is the kinematic viscosity, q is the discharge per unit length along the line sink, and ρ_0 is the undisturbed density. This restriction is never stated.

The most striking feature of Koh's solution is that if the far field for any discharge q, any density gradient, any viscosity, and any diffusivity is indeed described by his solution, then there can be no selective withdrawal at all in the usual sense, since the fluid enters the sink from all levels, its density being affected by conduction as it descends or ascends. Finally, we note that from the governing equations given by Koh [1966] one can immediately conclude that the horizontal velocity component u is even in the vertical coordinate y, and the deviation of the density from its *linear* profile is odd in y. This, together with the observations that the longitudinal conduction (of heat or salinity that affects the density) at any section with constant x contributes nothing to the heat flux and that the conduction at $\zeta = C$ and at $\zeta = -C$, C being any constant, cancel each other out, leads to the conclusion that the mean density of the fluid entering the line sink is that of the fluid at the level of the sink. This conclusion, which is of immediate practical importance, is not mentioned explicitly in Koh's paper [1966] and can be reached without his solution.

The same conclusion holds for slow axisymmetric flow into a point sink, for which Koh also gives a similarity solution. That solution, too, is valid only if gravity forces dominate viscous forces, thereby permitting a hydrostatic pressure distribution even at every large values of the viscosity.

2. Wood [1968] considered the withdrawal of a stratified fluid from a channel with a lateral contraction. He stated in the abstract that, when there are only two layers, "the conditions under which there is a flow in only one layer, and the depth in this flowing layer decreases continuously from its depth in the reservoir to its depth in the channel, give the maximum discharge that can be obtained with a flow only from this single layer." In the body of his paper one cannot find the calculation of this maximum discharge. Perhaps he considered the relevant flow a special case of two flowing layers. The paper did not deal with selective withdrawal squarely in that long before this maximum discharge is reached the fluid in the upper layer may already enter into the sink—a situation that Wood simply did not consider. His results, then, can be considered as providing an upper bound for the discharge from the lower layer (or upper layer, if the sink is at the upper boundary) when fluid is drawn from a reservoir by a channel, the contraction providing a control of the maximum allowable discharge in the lower layer, *if* no fluid in the upper layer enters the sink. For the cases of more than two layers or of a continuous stratification, he argued that there are more and more vertical control sections, gave a possible velocity distribution in the flowing layers of decreasing depth, characterized by a self-similarity, and showed that the depths at the section of maximum contraction are two-thirds of the depths far upstream, i.e., in the reservoir. But what causes the flow envisaged is left unexplained and unstated. Nor is the location of the sink in these cases unambiguously stated. As we have said, the location of the sink is crucial to the problem of selective withdrawal. Even if the location of the sink can be adequately specified, the results can at best be considered to provide an upper bound for the total discharge $Q(= Q_1 + Q_2)$ in the two flowing layers, *if* the overlying and underlying layers are stagnant. But even this is not clear, for the algebraic manipulations in the paper masked the physical basis for the calculation of Q_1 and Q_2 that he presented.

The 2/3 relationship is verified by his experiments. But the photographs show that the overlying layers were not exactly stagnant, for they were not horizontal. At any rate, it is not demonstrated theoretically that before the maximum Q in the supposed flowing layers is reached the supposed stagnant layers are really stagnant.

3. Lai and Wood [1975] consider three superposed fluid layers in a reservoir, of densities ρ_0, $\rho_0 + \Delta\rho_1$, and $\rho_0 + \Delta\rho_1 + \Delta\rho_2$. The top layer (of density ρ_0) is at rest, and fluids in the lower two layers are drawn at two points, say P_1 and P_2, downstream from a horizontal contraction. The point P_1 is presumably in the middle layer and the point P_2 in the bottom layer. The authors show that, for a given ratio of the discharges Q_1 and Q_2 (for the middle and bottom layers, respectively), the total discharge $Q = Q_1 + Q_2$ is limited, and furthermore some particular discharge ratios are not obtainable. The flow is assumed to be steady and gradual, so that the pressure distribution can be assumed hydrostatic. The Bernoulli equation is used for the flowing layers.

Apart from some ambiguities (for example, one curve in their Fig. 2, obtained by their theory, is ruled out as unrealistic), one weakness of this work is that the locations of points P_1 and P_2, let alone the sizes of the outlets for Q_1 and Q_2, are never specified, though these locations are very important and must crucially affect the limits of Q. Indeed, the central point in the problem of selective withdrawal is that what is drawn depends very much on the elevation of the outlet.

This weakness not withstanding, the possible flow profiles given as illustrations are interesting, and some experimental verification of these are desirable for the cases where the above-mentioned weakness does not invalidate the theory.

We may regard the work as open-channel hydraulics for two flowing layers and not as principally concerned with selective withdrawal. The criticism raised above can then be removed, Q_1 and Q_2 being what the actual situation allows.

Section 7

1. In a series of papers, Miles [1968a, -b] and Huppert and Miles [1969] gave solutions of Eq. (38) for a vertical, semicircular, and finally semi-elliptic barrier placed in a stratified flow of infinite extent. The flow is either confined between horizontal boundaries or in half space. In the latter case the density, being assumed linear, becomes first zero and then negative as the height increases. Correspondingly, the upstream horizontal velocity, being inversely proportional to the square root of the density, becomes first infinite, then imaginary(!). Thus Miles and Huppert joined the sizable number of distinguished researchers who have investigated *linear* density profiles extending to infinite height without explanation or comment.

Miles and Huppert rules out any solution with closed streamlines. This is logical, since the closed region does not originate upstream, where the form of the differential equation is determined. Yet in the photographs in Figs. 16b through f one sees regions of closed streamlines, and one wonders if ruling out the validity of solutions with regions of closed streamlines, logical as it is, may not also entail the sacrificing of solutions for more interesting flows.

2. Droughton and Chen [1971] adopted the method of generating barriers by singularities and compared the calculated results with their experimental results.

3. An indirect, numerical method for computing the flow field of a stratified fluid and the (long) obstacle form over which it flows is given by Lee and Su [1977]. The fluid is a superposition of homogeneous layers. Hence Boussinesq's approximation is not needed. Nonlinearity is preserved, but the pressure distribution is assumed hydrostatic. The results are useful.

4. The "unbounded" flow of a linearly stratified viscous but non-diffusive fluid over a vertical barrier was investigated by Janowitz for small Reynold numbers [1971] and for large Reynolds numbers [1973]. In both papers Oseen's approximation was used and waves were found in the lee. For the case of small Reynolds numbers an upstream wake was found. See Note 2 to Section 3 of Chapter 1. The usage of the word "unbounded" in the title of Janowitz's paper of 1973 is ill advised, since at sufficiently high elevations the density is negative, the undisturbed density profile being linear.

In both papers Fourier transforms are used for obtaining the solutions.

Sections 7 and 8

1. A long paper by McIntyre [1972] deals with the upstream influence of a slender body moving in a linearly stratified fluid. Boussinesq's approximation was used, and McIntyre found that the upstream influence depends, among other things, on whether the flow is bounded by rigid horizontal planes, and whether viscosity or time establishment is used to make the flow determinate. One wonders how for a linearly stratified fluid one can consider *unbounded* flows, or indeed use the Boussinesq approximation with confidence. The former objection can be removed by considering an exponentially stratified fluid. Also, as indicated in Section 6, barriers of height comparable to the total height of flow can change the upstream conditions drastically. For these high barriers there is yet no adequate analysis.

Section 13

1. If the Boussinesq approximation is used and the undisturbed density is assumed to vary exponentially or linearly with height, a solution for the equations of motion, continuity, and incompressibility for an inviscid and nondiffusive fluid, representing internal-wave motion of arbitrary amplitude, is possible, as pointed out by Drazin [1977], who attributed the solution to Rayleigh. Rayleigh, however, derived it only after linearization of the equations and hence did not know its validity for arbitrary amplitude.

Let x and z represent the horizontal and vertical coordinates, t be the time, k and m be the wave numbers for the directions of x and z, respectively, and ω be 2π times the frequency of the wave motion. Then the solution for the (two-dimensional) wave motion is

$$u = \omega k^{-1} a \cos \theta, \qquad w = -km^{-1} u,$$

where u and w are the velocity components in the directions of increasing x and z, respectively, a is a dimensionless amplitude, and

$$\theta = kx + mz - \omega t.$$

The pressure has, in addition to the hydrostatic part for the undisturbed density, the part $k^{-1}u$, and the deviation of the density from its undisturbed distribution is $N^2 m^{-1} a \sin \theta$. N here is the Brunt-Väisälä frequency defined in Chapter 2, which is constant if the undisturbed density varies exponentially with height, or can be taken to be constant, in consistency with the Boussinesq approximation, if the undisturbed density varies linearly with height.

Drazin then proceeded to study the stability of these waves, using the Floquet theory, since the solution cited above is periodic in time.

Janowitz [1968] found a solution for wave motion or arbitrary amplitude in a viscous, nondiffusive, linearly stratified fluid. Again the Boussinesq approximation is used. The prominent feature of the solution is that the motion is entirely vertical, though it depends on the horizontal coordinate. The waves attenuate, of course, as a result of viscosity.

Sections 13 and 14

1. Long nonlinear internal waves, including internal solitary waves, in channels of arbitrary cross section have been investigated by Shen [1968], who considered compressibility, and Grimshaw [1978], who considered an incompressible fluid. Both ignored viscosity, and both obtained the Korteweg-de Vries equation for weakly nonlinear effects.

Section 15

1. Zeytounian [1969a, -b] derived the equations governing steady flows of a stratified fluid. For two-dimensional flows, the equation of Dubreil-Jacotin [1935] and Long [1953b] is retrieved. But it is curious that in these papers, based on the dissertation of the author at the University of Paris, the name of Mme Dubreil-Jacotin, who was still on the faculty there, was not once mentioned. For three-dimensional steady flows the governing equations obtained are much the same as those of Yih [1967b]. These equations are difficult to solve without linearization. Zeytounian [1969b] solved them in their linearized forms, and although linear equations for steady three-dimensional stratified flows over barriers have been solved before [Scorer and Wilkinson, 1956; Crapper, 1959, 1962; and Drazin, 1961], Zeytounian's work is more systematic and, in any case, his calculations for the barrier form of a paraboloid of revolution give fascinating results. He also cited similar calculations by Trochu [1967] for the region of Cantal in the Massif Central of France and for the region near the basin of Arcachon, which, according to Trochu, predict the distributions of rain in these regions very well. The calculated rain distributions given by Trochu are very fascinating indeed.

Section 20

1. All further investigations since the work of Yih and Guha [1955] on internal hydraulic jumps are based on the assumption of hydrostatic pressure distribution assumed in Yih's analysis [Yih and Guha 1955]. The two-layer system was investigated by Mehrotra [1973] and Mehrotra and Kelly [1973] in search of new results. Mehrotra's definition of critical flow, whence ensue some of his results, is, in particular, not based on a firm foundation, so that a closer examination is necessary before his supercritical-to-supercritical jumps can be considered significant.

Su [1976] considered hydraulic jumps in a n-layer system and gave useful results. He also gave a layer approach to the equation of Dubreil-Jacotin [1935] and Long [1953b], not knowing that Yih [1957] had already done exactly the same thing 19 years earlier. The purpose of this layer approach is to give credence to the implied claim that hydraulic jumps in a fluid with continuous density distribution can be considered as limiting cases of jumps in a layered system. But this claim ignores a very special difficulty involving jumps in a fluid with continuous density distribution—that of turbulence at the jump and of the consequent mixing so characteristic and important for internal jumps. Indeed, the difficulty already exists for two layers of miscible fluids. But this difficulty is increased as the number of layers is increased, and is beyond one's power to cope with under the existing approach (there is essentially only one such method) when the density distribution is continuous. My judgment is that further progress on the study of hydraulic jumps in a stratified fluid of continuous density distribution will be made when one can find a way to describe how the flow of a fluid with continuous density stratification will behave in essentially a two-layer, three-layer, or n-layer way, with postjump slips or steep velocity gradients occurring only at the interfaces of these layers. The fluid then behaves essentially as a layered system, with interfacial mixing a nuisance to be dealt with as in an originally layered system. (This has not yet been done even for a layered system, for it involves some empirical information on the mixing between the layers at the jump.)

Section 21

1. It was implied in von Kármán's work [1940] that one can find a solution for potential flow over a current the surface of which is a streamline on which the pressure is the hydrostatic pressure in the heavier fluid. Benjamin [1968] pointed out that this is quite impossible if U is constant, since the *hydrodynamic* pressure (excluding the hydrostatic part) on the interface has a resultant of zero, and the hydrostatic forces on the interface cannot alone balance the hydrostatic pressure in the current at a cross section, say, far upstream. Benjamin concluded that there must be some separation and some turbulence near the top point of the gravity current, giving rise to some hydrodynamic drag. He is no doubt right. But so long as p_B does not differ much from the pressure at a point B' at the same elevation far upstream (i.e., to the left in Fig. 41a, where B' is not shown)—and it certainly does not—writing the Bernoulli equation between B' and A for the upper fluid and using the hydrostatic pressure relation between the same points for the lower fluid still gives von Kármán's formula for the speed of the gravity or density current, in spite of his oversight, which is worth mentioning but quite unimportant so long as one does not try to determine the actual shape of the interface and is interested only in determining U. The same conclusion can be drawn if the upper fluid is not infinite in extent.

2. Engelund and Pedersen [1973] studied the spread of a heated jet floating above and a cold liquid below, at low Richardson numbers. The question was taken up later by Engelund [1973, 1976] for slightly larger

Richardson numbers (less than 0.1). The integrated equations of continuity, mass conservation, and momentum were in both papers solved by using a similarity solution as a first approximation. Since the flow is turbulent, empirical constants obtained from experiments are used in the calculations in both papers.

Further experiments on surface buoyant jets by Rasmus Wiuff at the Technical University of Denmark also support Engelund's theory.

3. The speed of a density current of intermediate density advancing at the interface of two quiescent superposed fluids of densities greater and less than that of the current has been given, by the same approach as for the density current advancing in one quiescent fluid, by Engelund and Christensen [1969]. The case when the two superposed fluids have a relative velocity was considered by Moshagen [1972].

General

1. Abraham [1977] has analyzed data in the literature on interfacial shear of stratified flows, giving charts useful to hydraulic engineering.

2. The evolution of a horizontal jet in a stably stratified fluid has been investigated numerically by Peyret [1976], who included viscosity and thermal conductivity in the equations of motion and the heat equation. The Boussinesq approximation is used, and various Reynolds numbers and Froude numbers are considered. The Prandtl number considered is 10, near that for water.

3. List [1971] investigated momentum jets in a stratified fluid by neglecting the convective terms in the diffusion equation and the acceleration terms in the equation of motion (this statement cannot be found in the entire paper, but is presumably what the author meant when he spoke of "the linearized equations (13) and (14)"), and by adding singular terms for concentrated forces in the fluid. The accelerations must be infinite at the force singularities and very large nearby. Where this is true the Stokes approximations are not valid. For Stokes' solution for the falling sphere the singularities are all at the origin and hence within the sphere. For a sufficiently small Reynolds number the solution is then valid everywhere in the fluid. List's solution is valid where and only where the local Reynolds number and the local Péclet number are both very small. The axis of the jet is not required to be horizontal. A Fourier transform is used for the solution of the linear problem.

4. Turbulent mixing at a density interface was investigated by Rouse and Dodu [1955] by using an oscillating grid to create the turbulence. Their pioneering work was followed nearly two decades later by the work of Phillips [1972] and Kantha, Phillips, and Azad [1977] on turbulent entrainment. In this connection it should be mentioned that extensive and excellent work on turbulent mixing at an interface was done by Lofquist [1960], who investigated the velocity and stress distributions near an interface between stratified liquids in shear flow. His work will remain one of the few important sources of information on the subject.

5. Lumley [1972] gave a mathematical model for computing stratified turbulent flows. The equations for the first and second moments are closed by using forms asymptotically exact at infinite Reynolds number and/or Péclet number and by beginning with the exact expressions for the third-order moments as functionals of the second-order moments, expanding these functionals about the homogeneous and isotropic state, and retaining only the first-order terms in these expansions. Thus a closed set of 17 equations in 17 unknowns is obtained.

Chapter 4

HYDRODYNAMIC STABILITY

Le moindre vent qui d'aventure
Fait rider la face de l'eau . . .

La Fontaine

馮延巳有詞曰
風乍起吹皺一
池春水　李中
主戲之曰吹皺
一池春水干卿
底事　延巳對
曰未若陛下小
樓吹徹玉笙寒

Fung Yen-Sze, prime minister for the Middle King of South Tang, was the author of the poem which starts with*

The wind suddenly rises, and ruffles the surface
of the newly melted pond . . .

The king of South Tang, himself an accomplished poet, much appreciated this line, but, instead of praising it seriously, teased its author: "So the wind ruffles the water surface. What concern is that of yours?" Flattered by this subtle compliment, Fung returned it with "Not as good as Your Majesty's 'From the little pavilion wafted songs/that someone blew deep through his jade shêng/—his breath hardly warming the stone he touched.'"

1. INTRODUCTION

A liquid with density increasing in the direction of the vertical provides the most familiar example of hydrodynamic instability, of which gravity as well

* Chinese readers will recognize that the long-standing controversy concerning the poet's name is preserved in the versions given here. I have given the transliteration of what I believe to be the poet's correct name. I owe the calligraphy of the Chinese version to my friend Yuan Cheng Fung.

as the density stratification is the cause. A less familiar example is the instability of an atmosphere with entropy decreasing vertically upward. Certain stratified flows are, on the other hand, unstable although the stratification itself is stable. For such flows gravity is usually stabilizing, and the cause of instability is, broadly speaking, the vorticity distribution in the fluid. Sections 2 to 4 of this chapter are concerned with fluid configurations of which gravity is, or can be, the cause of instability. Sections 5 to 8 are devoted to flows in which gravity is a stabilizing factor, although in Section 5 the case of gravitational instability is included in a single general treatment.

It seems at first sight that for configurations which are unstable as a result of the action of gravity the cause of instability is perfectly obvious and therefore not very interesting. This is indeed true in a good many cases, in which only the rate of amplification of the disturbances need be determined. But if surface tension is taken into account in a system of superposed fluids, disturbances with a wavelength less than a critical one are stable, and the situation is already not so obvious. A more interesting role is played by thermal diffusivity, which can stabilize an otherwise unstable configuration. But what is the most interesting and subtle about diffusivities is that they can destabilize an apparently stable configuration.

As to the stability of stratified flows in a gravitational field, which will occupy our attention in the second half of this chapter, there is nothing obvious about it. In all the problems treated, the velocity distribution in the flow is of great importance, but it is the density variation (which can take the extreme form of a free surface) and the effect of gravity that distinguish them from the other problems of hydrodynamic stability. They constitute an important class of problems in modern hydrodynamics.

2. INSTABILITY OF TWO SUPERPOSED INVISCID FLUIDS IN A TUBE

The stability of two superposed fluids contained in a vertical circular tube will be considered first, because it has an interesting feature which is not obvious without detailed calculation.

The density of the lower fluid will be denoted by ρ and that of the upper fluid by ρ', and it will be assumed that $\rho' > \rho$, for otherwise there will be no instability. The cross section of the tube is a circle of radius a. The axis of the tube is assumed vertical, and is used as the z-axis of cylindrical coordinates (r, θ, z). The value of z increases in the direction of the vertical (opposite to that of gravity). The origin is taken at the center of the circular interface.

The effects of viscosities are neglected. The motion, assumed to have started from rest, is then irrotational. Let the velocity potentials for the two fluids be ϕ and ϕ'. The boundary conditions at the wall for both fluids are

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{and} \quad \frac{\partial \phi'}{\partial r} = 0 \quad \text{at} \quad r = a. \tag{1a}$$

The upper fluid is assumed to extend to $z = \infty$, and the lower fluid to $z = -\infty$. The case of finite depth can be similarly treated. Since the velocities of the fluids must vanish at $z = \pm\infty$,

$$\phi \to 0 \quad \text{as} \quad z \to -\infty \qquad \text{and} \qquad \phi' \to 0 \quad \text{as} \quad z \to \infty. \tag{1b}$$

Of course it would be sufficient to assume $\phi \to a$ constant as $z = -\infty$ and $\phi' \to$ another constant as $z \to \infty$. But there is no real loss of generality in assigning the value zero to these constants.

Keeping in mind (1) and (2), and the fact that ϕ and ϕ' must be harmonic (that is, must satisfy the Laplace equation), one can write down at once the forms of the velocity potentials*:

$$\begin{aligned} \phi &= C \exp(\sigma t + k_m z) \cos n\theta \, J_n(k_m r), \\ \phi' &= C' \exp(\sigma t - k_m z) \cos n\theta \, J_n(k_m r), \end{aligned} \tag{2}$$

in which J_n is the Bessel function of the nth order. The Neumann function $N_n(k_m r)$, being singular at $r = 0$, is not admissible. The eigenvalues k_m are roots of the equation

$$J_n'(kr) = \frac{dJ_n(kr)}{dr} = 0 \qquad \text{at} \quad r = a.$$

For $n = 0$, $ka = 3.8317, 7.0156, 10.1735, 13.3237$, etc. For $n = 1$, $ka = 1.84, 5.33, 8.53$, etc. For $n = 2$, the first eigenvalue for ka is 3.05, the second 6.70, and the third, 9.97.

The kinematic condition at the interface is that the vertical velocity component must be continuous, and this demands

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi'}{\partial z} = \frac{\partial \zeta}{\partial t} \qquad \text{at} \quad z = \zeta, \tag{3}$$

if ζ is the displacement of the interface. If the total pressure in the upper fluid is denoted by p_t' and that in the lower fluid by p_t, the dynamic boundary condition at the interface is

$$p_t - p_t' = -T\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{4}$$

in which T is the interfacial tension and R_1 and R_2 the principal radii, counted positive if the centers of curvature are *above* the interface. Now (4) must be satisfied already for the mean configuration, without any disturbances. The perturbation pressures p and p' must then satisfy the condition

$$p - p' = -T\left(\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2}\right), \tag{5}$$

* The forms are obtained on solving the Laplace equation in cylindrical coordinates by the method of separation of variables.

if terms of order higher than the first in ζ are neglected in a linear analysis.

With

$$\zeta = ae^{\sigma t}\cos n\theta\, J_n(kr), \tag{6}$$

equations (3) become

$$Ck = -C'k = \sigma a, \tag{7}$$

If equations (3) are applied at $z = 0$. This is of course a weakness of the analysis, since the undisturbed surface is already not flat.

The pressure p and p' can be found from the Bernoulli equations

$$\frac{p}{\rho} = -\frac{\partial\phi}{\partial t} + g\zeta \quad \text{and} \quad \frac{p'}{\rho'} = -\frac{\partial\phi'}{\partial t} + g\zeta. \tag{8}$$

Susbtitution (2), (6), and (8) into (5) yields, again with (5) applied at $z = 0$,

$$\rho(-\sigma C + ga) - \rho'(-\sigma C' + ga) = Tk^2 a. \tag{9}$$

Elimination of C, C', and a from (7) and (9) results in the equation

$$\sigma^2 = \frac{g(\rho' - \rho)k}{\rho + \rho'} - \frac{Tk^3}{\rho + \rho'}. \tag{10}$$

Thus, if

$$k^2 < \frac{g(\rho' - \rho)}{T},$$

the fluid is unstable. Now (10) could be written as

$$\sigma^2 = \frac{T(ka)}{(\rho + \rho')a^3}\left[\frac{g(\rho' - \rho)a^2}{T} - (ka)^2\right]. \tag{11}$$

Thus what mode is the most unstable depends on the value of $g(\rho' - \rho)a^2/T$. If it is very high, the most unstable mode will have a rather high eigenvalue of ka. If it is low, the most unstable mode will correspond to $ka = 1.84$ for $n = 1$. For small differences in density, the most unstable mode is therefore not axisymmetric—a rather interesting fact. However, incipient instability is always determined by

$$\left[\frac{g(\rho' - \rho)}{T}\right]^{1/2} a = \text{minimum value of } ka = 1.84.$$

Maxwell [1890, p. 587] tacitly assumed axial symmetry ($n = 0$), used 3.8317 instead of 1.84, and found his result to agree with the experimental results of Duprez [1851 and 1854] for incipient instability within less than two percent. Since Duprez used a fixed tube for his experiments, and changed the value of $(\rho' - \rho)/T$ by adding alcohol to the lower fluid (water) until it became lighter than the upper fluid (oil), it seems unlikely that the critical value he obtained for $(\rho' - \rho)/T$ could be four times too large. It appears that Maxwell was not aware of the fact that an asymmetric mode could be more unstable on theoretical grounds, or he would not have assumed axisymmetry in his analysis without explanation. If Duprez's experiments were free from errors, it seems that the axisymmetry of the undisturbed meniscus favored the gravest axisymmetric disturbance as the most unstable mode, and the analysis based on a flat interface must be applied with this in mind. At any rate a more accurate calculation seems desirable.

It is interesting, and somewhat surprising, that even the first mode characterized by the factor $\cos 2\theta$ is less stable than the gravest axisymmetric mode, since the first root of $J_2'(ka) = 0$ is $ka = 3.05$, which is less than 3.8317.

It remains to mention that (10) is valid for any deep container, whatever the cross-sectional shape, which only affects the eigenvalues of k. If the container has a large cross section so that the value

$$\frac{g(\rho' - \rho)d^2}{T} \gg 1,$$

in which d is a reference length, the interface will be flat, and the weakness of the analysis incurred in applying the interfacial conditions at $z = 0$ is not serious. In particular, (10) is exactly valid for two semi-infinite fluids with an interface. See Section 5.

The instability of a viscous fluid in a tube or between vertical walls heated from below was investigated by Hale [1951], Ostrach [1955], Taylor [1954], Yih [1959a], and Wooding [1960a]. Since the stability of a horizontal layer of fluid heated from below will be discussed in Section 4, we shall not enter into details for the somewhat similar case in which the fluid is bounded by vertical boundaries. Suffice it to say that for the circular tube the most unstable mode is the mode with a factor $\cos\theta$ (θ being the second of the cylindrical coordinates), as concluded by Taylor [1954] and shown by Yih [1959a], and that for the case of a fluid between two vertical walls, the most stable mode corresponds to infinite wavelength of the disturbance in the direction of the horizontal drawn in the plane of one of the walls [Wooding 1960a].

3. EFFECT OF VISCOSITY ON GRAVITATIONAL INSTABILITY

The problem of gravitational instability of an incompressible fluid with variable density and viscosity was considered by Chandrasekhar [1955a]. Since there is no mean flow, the solution of the mathematical problem is really quite straightforward, although the final secular equation can seldom be solved simply and never directly.

In Cartesian coordinates x_i ($i = 1, 2, 3$), and in the notation of Chapter 1, the Navier-Stokes equations are

$$\rho\left(\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}\right) = -\frac{\partial p}{\partial x_i} + \rho X_i + \frac{\partial}{\partial x_k}\left[\mu\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)\right], \tag{12}$$

in which μ is the viscosity. In the absence of any disturbances, the pressure distribution is simply hydrostatic. Thus the mean pressure $\bar{p}$ and the mean density $\bar{\rho}$ satisfy (12). If a disturbance is present, the velocity components will be u_i, and the pressure will be changed by an amount p' and the density by an amount ρ'. If the linearization process is applied, (12) becomes, when written out in full,

$$\begin{aligned} \bar{\rho}\frac{\partial u}{\partial t} &= -\frac{\partial p'}{\partial x} + \mu\,\nabla^2 u + \frac{d\mu}{dz}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \\ \bar{\rho}\frac{\partial v}{\partial t} &= -\frac{\partial p'}{\partial y} + \mu\,\nabla^2 v + \frac{d\mu}{dz}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \\ \bar{\rho}\frac{\partial w}{\partial t} &= -\frac{\partial p'}{\partial z} + \mu\,\nabla^2 w + 2\frac{d\mu}{dz}\frac{\partial w}{\partial z} - g\rho', \end{aligned} \tag{13}$$

in which the positive direction of z is the direction of the vertical. In obtaining (13),

$$\frac{d\bar{p}}{dz} = -g\bar{\rho}$$

has been utilized, and μ, the mean viscosity, has been tacitly assumed to be a function of z only. The coordinates x_i and velocity components u_i have been written in their customary forms.

The equation of incompressibility is, after linearization,

$$\frac{\partial \rho'}{\partial t} = -w\frac{d\bar{\rho}}{dz}, \tag{14}$$

and the equation of continuity is, exactly,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{15}$$

Actually, an equation

$$\frac{\partial \mu'}{\partial t} = -w\frac{d\mu}{dz}$$

should be added, in which μ' is the perturbation in viscosity. But μ' does not appear in the other equations. Hence this equation is used only to evaluate μ' after the other equations have been solved.

If all the perturbation quantities are assumed to have the exponential factor

$$\exp(\sigma t + imx + iny),$$

Eqs. (13) to (15) become

$$imp' = -\sigma\bar{\rho}u + \mu(D^2 - k^2)u + D\mu(imw + Du), \tag{16}$$

$$inp' = -\sigma\bar{\rho}v + \mu(D^2 - k^2)v + D\mu(inw + Dv), \tag{17}$$

$$Dp' = -\sigma\bar{\rho}w + \mu(D^2 - k^2)w + 2D\mu\,Dw - g\rho', \tag{18}$$

$$\sigma\rho' = -w\,D\bar{\rho}, \tag{19}$$

$$imu + inv = -Dw, \tag{20}$$

in which $k^2 = m^2 + n^2$, and $D = d/dz$. Elimination of u, v, p', and ρ' among Eqs. (16) to (20) produces

$$D\left\{\left[\bar{\rho} - \frac{\mu}{\sigma}(D^2 - k^2)\right] Dw - \frac{1}{\sigma}(D\mu)(D^2 + k^2)w\right\}$$
$$= k^2\left\{-\frac{g}{\sigma^2}(D\bar{\rho})w + \left[\bar{\rho} - \frac{\mu}{\sigma}(D^2 - k^2)\right]w - \frac{2}{\sigma}(D\mu)(Dw)\right\}, \qquad (21)$$

which is in fact the equation

$$D[-im\,(16) - in\,(17)] - k^2\,(18),$$

after (20) has been used, and ρ' has been replaced by $-w\,D\bar{\rho}/\sigma$, according to (19).

The boundary condition at a rigid boundary is that the velocity must be zero there. If the rigid boundary is horizontal, this means, by virtue of the equation of continuity,

$$w = 0 \qquad \text{and} \qquad Dw = 0 \qquad (22)$$

at that boundary. These conditions can also be imposed at infinity. At an interface the boundary conditions are:

(i) The velocity components are continuous,
(ii) the normal component of the stress is continuous;
(iii) the tangential component of the stress is continuous; (23)
(iv) $w = \sigma\zeta$, ζ being the displacement of the interface, assumed to have the same exponential factor.

The boundary condition (ii) contains ζ, because the quantity $\bar{p}$ on the interface depends on ζ. In case the surface is free, the boundary conditions are:

(i) The normal stress is zero, provided capillary effects are neglected;
(ii) the tangential stress is zero, provided the effects of variation of surface tension are neglected; (24)
(iii) $w = \sigma\zeta$.

The stress conditions in (23) and (24) need to be formulated mathematically. On a horizontal interface or free surface, the total shearing stress is equal to

$$\tau = \frac{1}{k}(m\tau_{zx} + n\tau_{zy}),$$

in which

$$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \mu(Du + imw)$$

and

$$\tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \mu(Dv + inw)$$

are the shearing stresses in a horizontal plane, in the directions of x and y, respectively. Thus

$$\tau = \frac{i\mu}{k}(D^2 + k^2)w,$$

and the continuity of τ at an interface demands

$$[\mu(D^2 + k^2)w]_u = [\mu(D^2 + k^2)w]_l, \tag{25}$$

in which the subscripts u and l indicate the upper and lower fluids, respectively. At a free surface $\mu_u = 0$, and (25) becomes, for the lower fluid,

$$(D^2 + k^2)w = 0. \tag{26}$$

Chandrasekhar [1955a] assumes a free surface to be a plane surface, which is, however, free from shear stress. Thus he has $w = 0$ on it, and hence also $D^2 w = 0$. Such an assumption does not seem to have a physical foundation. It is possible or even probable that when a liquid with a free surface *above* it undergoes internal convection, that free surface will remain so flat that w is essentially zero on it, so long as the density variation in the liquid is small compared with its mean density. But surely when the free surface is *below* the liquid $w = 0$ cannot possibly be even approximately true on it. His variational method, to be presented in this section, is therefore not generally valid without some modification.

The condition for the normal stress is slightly more troublesome, because it is necessary to find the condition on the displaced interface (or free surface), and not on a truly horizontal surface. The normal stress is

$$-(\bar{p} + p') + \mu\, Dw$$

on $z = \zeta$, hence is

$$-\bar{p}(0) - p' + g\bar{\rho}\zeta + 2\mu\, Dw = -\bar{p}(0) - p' + \frac{1}{\sigma}\bar{\rho}gw + 2\mu\, Dw,$$

in which $\bar{p}(0)$ is the mean pressure at the interface (or free surface), and p', w, and Dw are now evaluated at $z = 0$, where the mean interface is supposed to be. Now (16) and (17) give

$$p' = \frac{1}{ik^2}\{-\sigma\bar{\rho}(mu + nv) + \mu(D^2 - k^2)(mu + nv) + (D\mu)[ik^2 w + D(mu + nv)]\},$$

which, in virtue of (20), can be written as

$$p' = \frac{1}{k^2}[-\sigma\bar{\rho}\, Dw + \mu(D^2 - k^2)Dw + (D\mu)(D^2 + k^2)w].$$

The continuity of normal stress can then be expressed as

$$N_u = N_l, \tag{27}$$

in which

$$N = -k^2\left(\frac{1}{\sigma}\bar{\rho}gw + 2\mu\, Dw\right) - \sigma\bar{\rho}\, Dw + \mu\,(D^2 - k^2)Dw + (D\mu)\,(D^2 + k^2)w.$$

If the boundary is a free surface, (27) becomes, for the lower fluid,

$$N = 0. \tag{28}$$

Chandrasekhar [1955] obtained (27)—with $D\mu = 0$—for the interfacial condition of the special case of two homogeneous fluids, by integrating (21) across the interface. He did not take $w = 0$ on the interface. This is correct. Since a free surface is a special kind of an interface, w should not be zero on it either.

Chandrasekhar's variational method, based on $w = Dw = 0$ for a rigid boundary and $w = D^2w = 0$ on a free surface, and on the absence of any interface, is hence valid only for rigid boundaries and continuous stratification of the fluid. Since his method is very useful, it will be presented here, under the stated restrictions. If the planes $z = 0$ and $z = d$ are assumed to be rigid boundaries, and (21) is multiplied by σw and integrated (by parts whenever necessary) between 0 and d, the result

$$\sigma I_1 - \frac{g}{\sigma} I_2 = -I_3 \tag{29}$$

is obtained, in which

$$I_1 = \int_0^d \bar{\rho}[k^2w^2 + (Dw)^2]\, dz, \qquad I_2 = k^2 \int_0^d (D\bar{\rho})\, w^2\, dz,$$

$$I_3 = \int_0^d \left\{\mu[k^4w^2 + 2k^2\,(Dw)^2 + (D^2w)^2] + k^2\,(D^2\mu)\, w^2\right\} dz.$$

If w is subjected to a variation δw compatible only with the boundary conditions, σ will have a variation $\delta\sigma$, and (29) gives, after integration by parts,

$$-\left(I_1 + \frac{g}{\sigma^2} I_2\right)\delta\sigma = \sigma\,\delta I_1 - \frac{g}{\sigma}\,\delta I_2 + \delta I_3, \tag{30}$$

in which

$$\delta I_1 = 2\int_0^d \delta w\,[k^2\bar{\rho}w - D(\bar{\rho}\, Dw)]\, dz,$$

$$\delta I_2 = 2k^2 \int_0^d \delta w\,(D\bar{\rho})\, w\, dz,$$

$$\delta I_3 = 2\int_0^d \delta w\,\{k^4\mu w - 2k^2\, D(\mu\, Dw) + k^2\,(D^2\mu)\, w + D^2\,(\mu\, D^2w)\}\, dz.$$

Equation (30) can be written, after suitable combinations are made,

$$-\left(I_1 + \frac{g}{\sigma^2} I_2\right)\frac{\delta\sigma}{\sigma}$$

$$= 2\int_0^d \delta w \left\{k^2\left[\bar\rho w - \frac{\mu}{\sigma}(D^2 - k^2)w - \frac{g}{\sigma}(D\bar\rho)\, w - \frac{2}{\sigma}(D\mu)\,(Dw)\right]\right.$$

$$\left. - D\left[\bar\rho\, Dw - \frac{\mu}{\sigma}(D^2 - k^2)Dw - \frac{1}{\sigma}(D\mu)\,(D^2 + k^2)w\right]\right\} dz. \qquad (31)$$

Inspection of (31) reveals that the factor of δw in the integrand on the right-hand side is identical with (21). Thus the value of σ is stationary if the differential equation (21) is satisfied in the entire range, and vice versa, since δw is arbitrary aside from the fact that it must satisfy the boundary conditions. The usefulness of (31) consists in the realization that if a reasonable form for w is assumed, which satisfies the boundary conditions and agrees in general trend with what the true form of w must be, then the σ found from (29) must be near the true σ, since $\delta\sigma$ is nearly zero. Now (29) is quadratic in σ if the assumed form of w is independent of σ. Thus the approximate value of σ can be found easily from (29). Hide [1955b] used this approach to find the value of σ for a stratified fluid confined between two rigid boundaries.

As mentioned before, when free surfaces of viscosity discontinuities are present, Chandrasekhar's variation principle needs to be modified somewhat. The lower boundary will be assumed at $z = 0$ and upper boundary at $z = d$. (However, either or both of these boundaries may be at infinity, and the analysis will still be valid.) At $z = 0$, which is a rigid boundary, $w = Dw = 0$. The upper boundary may be rigid or free. If it is free, the boundary conditions (26) and (28) must be satisfied. There may be surfaces of fluid discontinuities, at which (25) and (27) must be satisfied. Multiplying (21) by w and integrating, by parts if necessary, between 0 and d, and utilizing all the boundary conditions wherever they are imposed, we get, after some rearrangements,

$$\sigma\int \bar\rho[k^2w^2 + (Dw)^2]\, dz - \frac{gk^2}{\sigma}\int (D\bar\rho)\, w^2\, dz$$

$$= -\int \mu\{[(D^2 + k^2)w]^2 + 4k^2(Dw)^2\}\, dz. \qquad (32)$$

Equation (32) is due to Selig [1964], and can be made the basis for a variation method. It replaces Chandrasekhar's formula, Eq. (29), when viscosity discontinuities are present, and it is somewhat preferable even if such discontinuities are absent, because the sign of $D^2\mu$ is now irrelevant. Thus (32) can be used to evaluate σ with suitable choices of w, subject only to the restriction of the boundary and interfacial conditions.

The case of two semi-infinite homogeneous viscous fluids with an interface was considered by Bellman and Pennington [1954] and Chandrasekhar [1955a], who treated essentially the same problem. The governing equation is, for both fluids,

$$\left[1 - \frac{\nu}{\sigma}(D^2 - k^2)\right](D^2 - k^2)w = 0,$$

the solutions of which are

$$w_1 = A_1 e^{kz} + B_1 e^{qz},$$
$$w_2 = A_2 e^{-kz} + B_2 e^{-q'z},$$

in which

$$q = \left(k^2 + \frac{\sigma}{\nu}\right)^{1/2}, \qquad q' = \left(k^2 + \frac{\sigma}{\nu'}\right)^{1/2},$$

with the primes indicating the upper fluid, and with q and q' so defined that their real parts are positive. Application of the interfacial conditions* results in the secular equation

$$-\left[\frac{gk}{\sigma^2}(\alpha - \alpha') + 1\right](\alpha' q + \alpha q' - k) - 4k\alpha\alpha'$$
$$+ \frac{4k^2}{\sigma}(\alpha\nu - \alpha'\nu')[(\alpha' q - \alpha q') + k(\alpha - \alpha')]$$
$$+ \frac{4k^3}{\sigma^2}(\alpha\nu - \alpha'\nu')^2(q - k)(q' - k) = 0, \tag{33}$$

in which

$$\alpha = \frac{\rho}{\rho + \rho'}, \qquad \alpha' = \frac{\rho'}{\rho + \rho'}.$$

For $\nu = \nu'$, Chandrasekhar was able to find a solution of (33) indirectly but exactly. For $\rho \geq \rho'$, he found that when the wave number k is greater than a critical value k_*, there are two modes of motion, one of which is quickly damped. The motion is oscillatory for $k < k_*$, and nonoscillatory for $k \geq k_*$. Hide [1955] applied (29) to find approximate solution of (33) for the general case of this problem. But, as we have seen, (29) is not valid when there are viscosity discontinuities. The stress conditions at the interface are not satisfied. Hence the agreement between his result for the case solved by Chandrasekhar and the latter's result is fortuitous, as pointed out by Reid [1960].

Equation (33) serves to determine the effect of the viscosities. If the Reynolds numbers for the disturbance, $\sigma/\nu k^2$ and $\sigma/\nu' k^2$, are both very large, (32) reduces to

$$\sigma^2 = gk\frac{\rho' - \rho}{\rho + \rho'}, \tag{34}$$

* Chandrasekhar obtained (27) by integration of (21) in the Stieltjes sense.

as in (10), since capillary effects are neglected here. Hence, when the disturbance Reynolds numbers are very large, the viscous effect is small. Of course we do not know σ *a priori*, but if we use the σ obtained from (34), and find the disturbance Reynolds numbers very large, we know (34) gives approximately the correct values of σ. If σ in (34) is plotted against k, the graph is a parabola passing through the origin, concave downward. Indeed, Chandrasekhar's exact solution for the case $\nu = \nu'$ shows that for small k the graph is in fact nearly parabolic, confirming our predictions here.

4. EFFECTS OF DIFFUSIVITIES ON GRAVITATIONAL INSTABILITY

The stability of a fluid heated from below was studied by Rayleigh [1916], Jeffreys [1926], Low [1929], Pellew and Southwell [1940], and others. The convection resulting from instability, however, was studied experimentally by Bénard much earlier [1901]. In this phenomenon, as in most others considered in this chapter, the effect of viscosity is to stabilize the fluid. Furthermore, the effect of thermal diffusivity is also stabilizing. This is understandable, because thermal diffusivity is an equalizing agent, which tends to harmonize the temperature (hence density) of a displaced fluid particle with that of its surroundings. However, if the fluid has a salinity gradient (or any equivalent gradient in concentration) as well as a temperature gradient, and if the fluid is warmer, saltier, and *lighter* on top, the effect of thermal diffusivity is to destabilize the fluid, because the harmonization of a displaced particle in temperature tends to displace the particle further from its original position. The destabilizing effect of thermal diffusivity is vividly evident in the "salt-fountain phenomenon" described by Stommel *et al.* [1956]. In their description an inner tube within the fluid container prohibits salinity diffusion while allowing thermal diffusion across its wall, thus maintaining a fresh water current going up from the bottom of the tube, becoming warmer and lighter as it goes. Since thermal diffusivity of a saline solution is greater than its salinity diffusivity, the phenomenon can occur even in the absence of the inner tube. This section is devoted to a discussion of it.

Consider, after Stern [1960b], the stability of a horizontal layer of water (or some other liquid) heated from below. The mass concentration of a chemical dissolved in the liquid is maintained at c_0 at the lower boundary, and at c_1 at the upper boundary. The temperatures at the two boundaries are maintained at T_0 and T_1 respectively, with $T_1 > T_0$. The maintenance of the boundary temperatures can be accomplished by using copper plates as the boundaries.

The maintenance of wall concentrations is more difficult to achieve. No special care need be taken of the lower plate. The upper plate should be covered with a fine copper screen which holds the chemical in the form of powder or small crystals in a *thin* layer between itself and the copper plate. The concentration at the upper boundary will be the saturation concentration c_1 at the temperature T_1. After the steady state has been reached the concentration at the lower plate will be the saturation concentration c_0 at temperature T_0. Since $T_1 > T_0$, for most chemicals $c_1 > c_0$. The chemical continues to precipitate* on the lower plate, as long as the chemical at the upper boundary is not exhausted. So long as the mesh size of the screen at the upper boundary is very small compared with the spacing d of the two boundaries, it should not affect the size of the convection cells.

The Navier-Stokes equations of motion are

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho X_i + (\lambda + \mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_\alpha}{\partial x_\alpha}\right) + \mu \nabla^2 u_i, \tag{35}$$

in which λ and the viscosity μ are assumed to be constant. The quantity λ is related to μ and the volume viscosity μ_v by

$$\lambda + \tfrac{2}{3}\mu = \mu_v .$$

If the direction of increasing x_3 is the direction of the vertical, the body force per unit mass is

$$(X_1, X_2, X_3) = (0, 0, -g). \tag{36}$$

The equation of heat transfer is

$$\frac{\partial T}{\partial t} + \frac{\partial (Tu_\alpha)}{\partial x_\alpha} = \frac{DT}{Dt} + T\frac{\partial u_\alpha}{\partial x_\alpha} = \kappa \nabla^2 T, \tag{37}$$

in which T is the temperature and κ is the thermal diffusivity. Viscous dissipation, which is a second-order quantity in the velocity, is neglected. The diffusion equation is

$$\frac{\partial c}{\partial t} + \frac{\partial (cu_\alpha)}{\partial x_\alpha} = \frac{Dc}{Dt} + c\frac{\partial u_\alpha}{\partial x_\alpha} = \kappa' \nabla^2 c, \tag{38}$$

in which c is the concentration of the chemical (in mass per unit volume), and κ' is the mass diffusivity. The variation of density with temperature and concentration is given by

$$\rho = \rho_0[1 - \alpha(T - T_0) + \alpha'(c - c_0)], \tag{39}$$

in which α is the coefficient of expansion, and $\alpha' = 1/\rho_0$. The variation of ρ with pressure is very small for a liquid, and is negligible. Care must be taken in writing the equation of continuity. If u_i is defined to be the ratio of the ith component of the total momentum to the total mass of all molecules in a fluid particle, be they molecules of the solvent or the solute, the equation of continuity is still

$$\frac{D\rho}{Dt} + \rho\frac{\partial u_\alpha}{\partial x_\alpha} = 0. \tag{40}$$

* Precipitation in the interior of the fluid is another form of instability. Only the case in which this does not occur is studied here.

But then $\partial u_\alpha/\partial x_\alpha$ is no longer the rate of volume expansion per unit volume. If u_i is defined to be the unweighted average of the ith component of the velocity of all the molecules, as usual, then diffusion of the solute has to be taken into account, and, for a dilute (even if saturated) solution such as under consideration here, the equation of continuity takes the form

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_\alpha}{\partial x_\alpha} = \kappa' \nabla^2 c. \tag{41}$$

The right-hand side is the rate of increase of mass per unit volume due to solute diffusion. From (38) and (41) it follows that

$$\frac{D}{Dt}(\rho - c) + (\rho - c)\frac{\partial u_\alpha}{\partial x_\alpha} = 0, \tag{42}$$

which states that diffusion of the solvent itself is negligible for dilute solutions. Equations (39) through (42) can be put on a firmer basis by molecular-theoretic considerations.

The mean configuration is described by

$$\left.\begin{aligned} \bar{T} &= T_0 + \beta x_3, \\ \bar{c} &= c_0 + \beta' x_3, \\ \bar{\rho} &= \rho_0[1 - (\alpha\beta - \alpha'\beta')x_3], \\ \frac{\partial \bar{p}}{\partial x_3} &= -g\bar{\rho}, \qquad \text{and} \qquad u_i = 0, \end{aligned}\right\} \tag{43}$$

in which

$$\beta = \frac{T_1 - T_0}{d}, \qquad \beta' = \frac{c_1 - c_0}{d}. \tag{44}$$

For a small perturbation from the mean configuration,

$$T = \bar{T} + T', \qquad c = \bar{c} + c', \tag{45}$$

in which T' and c' are the perturbation quantities, as are the velocity components u_i. (Note that α' and β' are not perturbation quantities.) First, (42) will be examined, to see how great the rate of volume dilatation is. Substituting (39) into (42), we see that

$$-\alpha\rho_0 \frac{DT}{Dt} + (\rho - c)\frac{\partial u_\alpha}{\partial x_\alpha} = 0. \tag{46}$$

Now the coefficient of expansion α is very small compared with unity for most liquids and solutes. Since DT/Dt is of the first order in the perturbation quantities (because $\bar{T}$ is independent of t), we can assume

$$\frac{\partial u_\alpha}{\partial x_\alpha} = 0. \tag{47}$$

This is not to say that there is no volume expansion. It only says that it is small, and can be taken to be zero as far as continuity is concerned. On the other hand, the term ρX_3 in (35) represents the body force. In it the effect of thermal expansion must be taken into account, however small this expansion is, because body force is the motive force of convection.

With (47) and the assumption that $c - c_0$ is everywhere small compared with ρ_0, the governing equations are greatly simplified. Equations (35), in linearized form, now become, after (43) has been utilized,

$$\frac{\partial u_i}{dt} = (0, 0, g\alpha T' - gc'/\rho_0) - \frac{1}{\rho_0}\frac{\partial p'}{\partial x_i} + \nu \nabla^2 u_i, \tag{48}$$

in which $\nu = \mu/\rho$ is the kinematic viscosity. Equations (37) and (38) become

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) T' = -\beta u_3 \tag{49}$$

and

$$\left(\frac{\partial}{\partial t} - \kappa' \nabla^2\right) c' = -\beta' u_3 . \tag{50}$$

The third equation in (48), operated on by ∇^2, is

$$\frac{\partial}{\partial t} \nabla^2 u_3 = -\frac{1}{\rho_0}\frac{\partial}{\partial x_3} \nabla^2 p' + g\alpha \nabla^2 T' - \frac{g}{\rho_0} \nabla^2 c' + \nu \nabla^2 \nabla^2 u_3 . \tag{51}$$

On the other hand, taking the divergence of (48), and differentiating the result with respect to x_3, we have

$$0 = \frac{\partial^2}{\partial x_3^2}\left(g\alpha T' - \frac{g}{\rho_0} c'\right) - \frac{1}{\rho_0}\frac{\partial}{\partial x_3} \nabla^2 p'. \tag{52}$$

From (51) and (52) it follows that

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \nabla^2 u_3 = g\alpha \nabla_2^2 T' - \frac{g}{\rho_0} \nabla_2^2 c', \tag{53}$$

in which

$$\nabla_2^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^3} .$$

Since u_1, u_2, and u_3 must vanish on the boundaries,

$$u_3 = 0 \quad \text{and} \quad \frac{\partial u_3}{\partial x_3} = 0 \quad \text{at} \quad x_3 = 0 \quad \text{and} \quad d, \tag{54}$$

the second condition being a consequence of (47). The boundary conditions for T' and c' are as explained in detail in the second paragraph of this section, and are

$$T' = 0 \quad \text{and} \quad c' = 0 \quad \text{at} \quad x_3 = 0 \quad \text{and} \quad d. \tag{55}$$

Thus, the governing differential system consists of (49), (50), (53), (54), and (55).

Cross differentiation of the first two equations in (48) produces

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right)\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = 0.$$

If we take

$$\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0,$$

so that the flow in the plane of x_1 and x_2 is irrotational, we have

$$u_1 = \frac{\partial \phi}{\partial x_1}, \qquad u_2 = \frac{\partial \phi}{\partial x_2},$$

and

$$\nabla_2^2 \phi = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = -\frac{\partial u_3}{\partial x_3}. \tag{56}$$

This equation can be solved with the condition $\phi = 0$ at $x_3 = 0$ and d, and the conditions for u_1 and u_2 at the boundaries are then satisfied.

The technique of solving differential system is essentially that of separation of variables. With

$$(x, y, z) = \left(\frac{x_1}{d}, \frac{x_2}{d}, \frac{x_3}{d}\right), \qquad \tau = \frac{\kappa}{d^2}\, t,$$

we may try

$$\begin{aligned} u_3 &= (\kappa/d) f(x, y)\, w(z)\, e^{\sigma\tau}, \\ T' &= (T_1 - T_0) f(x, y)\, \theta(z)\, e^{\sigma\tau}, \\ c' &= (c_1 - c_0) f(x, y)\, \gamma(z)\, e^{\sigma\tau}. \end{aligned} \tag{57}$$

For the variables to be separable, $f(x, y)$ must satisfy

$$f_{xx} + f_{yy} + a^2 f = 0, \tag{58}$$

in which a^2 is a constant arising from the separation of variables. For rectangular cells

$$f(x, y) = \cos\frac{n_1 \pi x_1}{L_1} \cos\frac{n_2 \pi x_2}{L_2},$$

provided

$$\left(\frac{n_1}{L_1}\right)^2 + \left(\frac{n_2}{L_2}\right)^2 = \left(\frac{a}{\pi d}\right)^2.$$

If the container is large enough, in the case of heat diffusion alone, Bénard [1901] found the pattern (for the case of a free upper surface) to be hexagonal.

Christopherson [1940] showed that the solution of (58) representing hexagonal convection is*

$$f(x, y) = \text{constant} \cdot \left\{ \cos \frac{2n\pi d}{3L}(\sqrt{3}\,x + y) + \cos \frac{2n\pi d}{3L}(\sqrt{3}\,x - y) + \cos \frac{4n\pi d y}{3L} \right\},$$

with

$$\frac{aL}{d} = \frac{4n\pi}{3}.$$

Thus a is a sort of wave number. Substitution of (57) into (49), (50), and (53) then yields

$$\begin{aligned} [\sigma - (D^2 - a^2)]\theta &= -w, \\ [\sigma - k(D^2 - a^2)]\gamma &= -w, \\ [\sigma/Pr - (D^2 - a^2)](D^2 - a^2)w &= -Ra^2\theta + R'a^2\gamma, \end{aligned} \tag{59}$$

in which

$$k = \frac{\kappa'}{\kappa}, \qquad Pr = \frac{\nu}{\kappa}, \qquad R = \frac{g\alpha\beta d^4}{\kappa\nu}, \qquad R' = \frac{g\beta' d^4}{\rho_0 \kappa\nu}, \qquad D = \frac{d}{dz}. \tag{60}$$

The boundary conditions are

$$\begin{aligned} w(0) = 0 = w(1), \qquad & w'(0) = 0 = w'(1), \\ \theta(0) = 0 = \theta(1), \qquad & \gamma(0) = 0 = \gamma(1). \end{aligned} \tag{61}$$

In the absence of mass diffusion, Pellew and Southwell [1940] proved that if the imaginary part (σ_i) of σ is not zero, the real part (σ_r) of it must be negative. Hence σ_i is zero for nonnegative values of σ_r. Thus, for neutral stability ($\sigma_r = 0$) $\sigma = 0$. This is called the principle of exchange of stabilities. The proof of Pellew and Southwell does not work in general in the present case, although it does work for $\kappa = \kappa'$. Therefore it is possible in some cases that σ_i is not zero when σ_r is zero. In case the principle of exchange of stabilities is valid, for neutral stability equations (59) become

$$\begin{aligned} (D^2 - a^2)\theta &= w, \\ (D^2 - a^2)\gamma_1 &= w, \\ (D^2 - a^2)^2 w &= a^2(R\theta - R'\gamma_1/k), \end{aligned} \tag{62}$$

* Professor Otto Laporte pointed out to the author that the *fundamental* pattern represented by $f(x, y)$ is really equilateral-triangular, only hexagonal in a composite way.

with $\gamma_1 = k\gamma$. Inspection of (62) and the boundary conditions reveals that $\theta = \gamma_1$, and (62) can be replaced by

$$(D^2 - a^2)\theta = w,$$
$$(D^2 - a^2)^2 w = a^2(R - R_1)\theta, \tag{63}$$

in which

$$R_1 = \frac{R'}{k} = \frac{g(c_1 - c_0)d^3}{\rho_0 \kappa' \nu}.$$

The solution of (63) with the boundary conditions for w and θ in (61) have been given exactly by Pellew and Southwell. The critical value for $R_1 - R$ is 1708. Here is the interesting situation. Even if $\alpha(T_1 - T_0)\rho_0$ is larger than $c_1 - c_0$, so that the fluid in the undisturbed condition is lighter at the top than at the bottom, $R_1 - R$ may still exceed 1708, because κ' may be smaller than κ, as in the case of the saline solution. The instability in that case is due to the greater facility with which a displaced particle harmonizes with its new surroundings in temperature than in concentration. For a particle displaced from above, this results in its becoming heavier than its environment and favors its further downward movement. For a particle displaced from below, this results in its becoming lighter than its environment and favors further upward movement. In either case the fact that the thermal diffusivity is greater than the mass diffusivity favors instability. In other words, thermal diffusivity is destabilizing in the case under consideration, whereas mass diffusivity is stabilizing. If the solubility of a substance varies greatly with temperature, it is possible to have a layer of its solution heated from below and saturated everywhere which is lighter at the top than at the bottom. In that case mass diffusivity is destabilizing, whereas thermal diffusivity is stabilizing.

If the origin of z is taken midway between the plates, the boundary conditions for (63) become

$$w(\pm\tfrac{1}{2}) = 0, \qquad w'(\pm\tfrac{1}{2}) = 0, \qquad \theta(\pm\tfrac{1}{2}) = 0.$$

A glance at (63) and these boundary conditions reveals that w and θ can be both odd or both even in z. The most unstable mode is for the even w and θ which correspond to a single cell between the boundaries. Since even disturbances and odd disturbances have different rates of damping or amplification, the question arises as to the behavior of a disturbance which is neither even nor odd. The answer is that this disturbance can be split into two, one of which is even, the other odd, and these have different rates of damping or amplification. This is related to the fact that the secular equation for a general disturbance can be factored into two equations, one for the even part and the other for the odd part of the disturbance. In the present case, the secular equation is the vanishing of a 6-by-6 determinant, because (63) can be reduced to a single equation of the sixth order. This determinant can be factored into two 3-by-3 determinants, one on the upper left corner and the other on the lower right corner, while the elements in the remaining positions are reduced to zero. The details provide a short and interesting exercise.

Chandrasekhar [1954d, -f] proposed two methods for solving (63) approximately, both of which are also used in the theory of elasticity. The first is a variational method similar to that presented in Section 3. The second, also proposed in a slightly different form by Malurkar and Srivastava [1937], is much more powerful, because it applies even to the case for which exact solutions are not possible. For this reason its procedure will be briefly described here, with (63) as an example. First, assume

$$\theta = \sum_{n=1}^{\infty} A_n \sin n\pi z, \tag{64}$$

which satisfies the boundary conditions for θ. Then substitute (64) in the second equation in (63), and solve the nonhomogeneous equation for w, arranging to have the boundary conditions for w satisfied by w_n (for each value of n) in the series solution of the form

$$w = \sum_{n=1}^{\infty} A_n\, w_n(z). \tag{65}$$

Now substitute (65) in the first equation in (63) and sort out the Fourier coefficients of $\sin n\pi z$, in the usual way (that is, multiplying by $\sin m\pi z$ and integrating from 0 to 1). Equate these coefficients to zero, and obtain infinitely many linear equations in A_1, A_2, etc. The determinant of the coefficients of A_n must vanish if the A's are not to be all zero, which they must not be. This determinant contains $R_1 - R$ and a^2, and its vanishing provides the required secular equation. Chandrasekhar has shown that even the vanishing of the upper left element (first element) of the determinant gives a good approximation to the true relationship between a^2 and $R_1 - R$ (corresponding to his R), and that convergence is fast as the order of the determinant used is increased. (The determinant used is always an n-by-n determinant at the upper left corner of the infinite determinant.)

5. HELMHOLTZ INSTABILITY, OR INSTABILITY OF TWO SUPERPOSED INVISCID FLUIDS

It was Helmholtz [1868] who first considered the stability of the vortex sheet at the interface of two superposed semi-infinite fluids flowing with different velocities. His work was followed by that of Kelvin [1871]. The density of the lower fluid will be denoted by ρ and that of the upper fluid by ρ'. The velocities of the two fluids are assumed to be horizontal in direction, and are denoted by U and U', respectively. If the effects of viscosity of the fluids are neglected, and the perturbed flow is assumed to be irrotational, the velocity potentials of the two fluids can be written, respectively, as

$$\phi = Ux + \phi_1, \qquad \phi' = U'x + \phi_1', \tag{66}$$

in which x is measured in the direction of the mean velocities, and all the ϕ's satisfy the Laplace equation.

If the direction of increasing z is the direction of the vertical, and ζ is the displacement of the interface in the z-direction, the kinematic conditions to be satisfied at $z = 0$ are

$$\frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x} = \frac{\partial \phi}{\partial z}, \qquad \frac{\partial \zeta}{\partial t} + U' \frac{\partial \zeta}{\partial x} = \frac{\partial \phi'}{\partial z}, \tag{67}$$

in which quadratic terms in ζ, ϕ_1, and ϕ_1' are neglected. Other boundary conditions for ϕ_1 and ϕ_1' are, without loss of generality,

$$\phi_1 \to 0 \quad \text{as} \quad z \to -\infty, \qquad \text{and} \qquad \phi_1' \to 0 \quad \text{as} \quad z \to \infty,$$

which guarantee the vanishing of velocities at $z = \pm\infty$.

The dynamic boundary condition at the interface is (5), with the Laplacian written in Cartesian coordinates. If terms of higher order than the first in ζ are neglected, the condition is

$$p - p' = -T\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right), \tag{68}$$

in which y is measured in a horizontal direction normal to that of x. Since the flow is assumed to be irrotational, the Bernoulli equation can be used to evaluate p. The linearized form of it is

$$\frac{p}{\rho} = -\frac{\partial \phi_1}{\partial t} - U \frac{\partial \phi_1}{\partial x} - gz, \tag{69}$$

and a similar formula gives p' in terms of ϕ_1'. Applying the formulas for p and p' to (68) at $z = \zeta$, one has

$$\rho\left(-\frac{\partial \phi_1}{\partial t} - U \frac{\partial \phi_1}{\partial x} - g\zeta\right) - \rho'\left(-\frac{\partial \phi_1'}{\partial t} - U' \frac{\partial \phi_1'}{\partial x} - g\zeta\right) = -T\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right). \tag{70}$$

If the perturbation is assumed to be periodic in x and y, the appropriate forms for ϕ_1, ϕ_1' and ζ are

$$\phi_1 = C \exp\,[kz + i(\sigma t - mx - ny)],$$

$$\phi_1' = C' \exp\,[-kz + i(\sigma t - mx - ny)],$$

$$\zeta = a \exp\,[i(\sigma t - mx - ny)].$$

Note that $i\sigma$ in this section correspond to the σ in the previous sections in this chapter. It is evident that ϕ_1 and ϕ_1' satisfy the Laplace equation if

$$k^2 = m^2 + n^2,$$

and that the boundary conditions at $z = \pm\infty$ are satisfied. Equations (67) and (70) now become

$$i(\sigma - mU)a = kC, \qquad i(\sigma - mU')a = -kC',$$

and

$$\rho[i(mU - \sigma)C - ga] - \rho'[i(mU' - \sigma)C' - ga] = Tk^2 a.$$

Elimination of C, C', and a yields

$$\rho(\sigma - mU)^2 + \rho'(\sigma - mU')^2 = gk(\rho - \rho') + Tk^3,$$

or

$$\frac{\sigma}{m} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \left\{\frac{gk}{m^2}\frac{\rho - \rho'}{\rho + \rho'} + \frac{Tk^3}{m^2}\frac{1}{\rho + \rho'} - \frac{\rho\rho'}{(\rho + \rho')^2}(U - U')^2\right\}^{1/2}. \tag{71}$$

If the disturbance is two-dimensional, $n = 0$ and $m = k$, and (71) becomes

$$\frac{\sigma}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \left\{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} + \frac{Tk}{\rho + \rho'} - \frac{\rho\rho'}{(\rho + \rho')^2}(U - U')^2\right\}^{1/2}, \tag{72}$$

which is in effect given in Lamb [1945, p. 373 and p. 462]. We shall first discuss (72), before returning to (71).

The right-hand side of (72) is the phase velocity of the disturbance. The first term on the right-hand side is a weighted (by the density) mean velocity of the two streams. Relative to this velocity are waves traveling with velocities $\pm c$, if

$$c^2 = c_0^2 - \frac{\rho\rho'}{(\rho + \rho')^2}(U - U')^2$$

is positive, in which c_0 is the wave velocity in the absence of currents, given by

$$c_0^2 = \frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} + \frac{Tk}{\rho + \rho'}.$$

Now c_0^2 is a function of k, and attains its minimum value of $[2Tg(\rho - \rho')/(\rho + \rho')^2]^{1/2}$ at

$$k^2 = \frac{g(\rho - \rho')}{T} = k_{cr}^2,$$

if $\rho > \rho'$. For $k < k_{cr}$, c_0 decreases with k, so that the group velocity

$$\frac{d(kc_0)}{dk} = c_0 + k\frac{dc_0}{dk}$$

is less than c_0, and the waves are principally gravitational in nature. For $k > k_{cr}$, the reverse is true, and the waves are principally capillary in nature. If a two-dimensional obstacle is placed in a stream with $U = U'$, the capillary waves created will appear only upstream, whereas the gravitational waves will appear only downstream, provided the stream velocity U is greater than the minimum value of c_0 corresponding to k_{cr}. If $U <$ minimum c_0, no standing waves will result. For details, see Lamb [1945, pp. 464–468].

Now of course c^2 need not be positive. In fact, if $\rho < \rho'$, even c_0^2 need not be positive. If c^2 is not positive, the disturbance is not stable. This is the case if $\rho > \rho'$ and

$$\frac{\rho\rho'}{(\rho + \rho')^2}(U - U')^2 > \text{minimum value of } c_0^2.$$

If $\rho < \rho'$, the disturbance is unstable for those values of k for which

$$Tk < \frac{\rho\rho'}{\rho + \rho'}(U - U')^2 + \frac{g}{k}(\rho' - \rho),$$

that is, for those (positive) values of k less than

$$\tfrac{1}{2}(A + \sqrt{A^2 + 4g(\rho' - \rho)/T}),$$

in which

$$A = \frac{\rho\rho'}{(\rho + \rho')T}(U - U')^2.$$

We shall now return to (71). If it is multiplied by m/k, it becomes

$$\frac{\sigma}{k} = \frac{m}{k}\frac{\rho U + \rho' U'}{\rho + \rho} \pm \left\{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} + \frac{Tk}{\rho + \rho'} - \frac{\rho\rho'}{(\rho + \rho')^2}\frac{m^2}{k^2}(U - U')^2\right\}^{1/2}.$$

This is identical with (72) if the U and U' in (72) are replaced by mU/k and mU'/k, respectively. Now m/k is the cosine of the angle between the main stream and the direction of propagation of the *two*-dimensional disturbance with the exponential factor

$$\exp\,[i(\sigma t - mx - ny)].$$

Since a three-dimensional disturbance is the superposition of two two-dimensional disturbances with exponential factors

$$\exp\,[i(\sigma t - mx - ny] \qquad \text{and} \qquad \exp\,[i(\sigma t - mx + ny)],$$

respectively, and otherwise identical, the stability or instability of the stream with respect to three-dimensional disturbances can be inferred from the stability or instability, with respect to two-dimensional disturbance, of a stream with velocities reduced by the factor m/k. This is so because a linear approach is used, and superposition is permitted. The conclusion was reached by Squire [1933] for a viscous liquid flowing between rigid planes. The present interpretation is due to Lin [1954; see also p. 77, 1955]. Yih [1955] has used Lin's approach to show that the stability or instability of parallel stratified flows of a viscous fluid down a plane with slope s, longitudinal pressure gradient $\partial p/\partial x$, and Reynolds number R with respect to three-dimensional disturbances can be inferred from the stability or instability of flows of the same stratification down a plane with slope s', pressure gradient $\partial p'/\partial x'$, and Reynolds number R', with respect to two-dimensional disturbances, provided

$$kR' = mR, \qquad k \sin \beta' = m \sin \beta, \qquad k \frac{\partial p'}{\partial x'} = m \frac{\partial p}{\partial x}, \tag{73}$$

with β and β' being the angles of inclination, such that

$$s = \tan \beta, \qquad s' = \tan \beta'. \tag{73a}$$

The discussion so far has been based on the assumption that the two fluids move in the same direction. If they do not, a coordinate system moving with one fluid can be used. With respect to this system the other fluid moves with the vector difference of the two velocities, and the problem is reduced to the one just discussed. The treatment for finite depths is similar to the one given here.

It remains to mention only that in the absence of currents (71) or (72) gives (10) again, which indicates instability if

$$\rho < \rho' \qquad \text{and} \qquad k^2 < \frac{g(\rho' - \rho)}{T}.$$

(Note that $i\sigma$ in this section is σ in (10).)

6. WAVE GENERATION BY WIND

The generation of waves in water by wind blowing over its surface is a geophysical phenomenon of considerable importance. Many attempts have been made to explain it, but Ursell, writing on this subject in the Surveys of Mechanics in 1956, concluded that "the present state of our knowledge is profoundly unsatisfactory." In the following six years, a series of papers by Miles [1957, 1959a, -b, 1960, 1962] supplied the theory, which has been found to be in adequate agreement with observations [Longuet-Higgins, Cartwright, and Smith, 1962]. Before sketching out Miles' theory, other mathematical models adopted to explain the phenomenon will be briefly summarized.

(1) The Helmholtz model, as presented in Section 5, is based on the instability of a vortex sheet. For air blowing on water, T is about 74 dynes per centimeter at 20° C, $\rho'/\rho = 0.00129$, and (72) gives the wind speed for incipient instability to be 646 cm/sec, or about 14.5 miles per hour, if the water speed is zero. At wind speeds as high as this, the mechanism of Helmholtz stability may actually contribute to the generation of waves by wind. One weak point in the Helmholtz model is the rather unrealistic velocity discontinuity at the interface and the uniform velocity of the wind. Miles [1959b] improved the theory by using the logarithmic profile for the wind. This profile obviously cannot be applied right at the interface. Miles applied it at a small distance above it, and indicated that the error committed is small. However, the most important and least obvious mechanism of wave generation is not the Helmholtz mechanism, but the mechanism proposed by Miles in 1957, whose theory is the main concern of this section.

Of course the stability of two streams with an interface can be investigated rather exactly if the flow is entirely laminar, as done by Wuest [1949] and Lock [1954]. For actual wave generation by wind in nature, however, it is more realistic to assume the wind to be turbulent.

(2) Jeffreys [1924 and 1925a] assumed a wind stress with a component in phase with wave slope rather than with wave height, and considered this component to be a consequence of separation at wave crests, which were supposed to have a "sheltering" effect. Jeffreys did not say how this effect could be evaluated, and at any rate Ursell [1956] has shown that the wave velocity may be and often is higher than the air velocity near the interface, in which case separation on the leeward side of the crests cannot be expected. (See Lighthill [1962].) Actually, Jeffrey's sheltering model does have one point in common with Miles' theory [1957], in that in both a wind stress in phase with the wave slope rather than the wave height is considered to be responsible for wave generation.

(3) Eckart [1953] assumed a wind structure characterized by gusts of prescribed durations in time and extent in space, and investigated the effect of this wind on wave generation. Although the air movement may closely approximate that in a storm, Eckart's model is primarily for wave generation by gusts, and is thus quite different from that of Jeffreys or Miles. Phillips [1957] proposed a mechanism of wave generation based on resonance of water waves with that component of the pressure fluctuations in the air that travels with the wave velocity of water. However, Longuet-Higgins [1962] showed that the mean square of the pressure fluctuations is at least two orders of magnitude smaller than what Phillips estimated. However, the small waviness produced by Phillips' mechanism may provide the initial waviness needed in the stability theory of Miles [1957].

We shall now present Miles' theory of 1957. As already mentioned, this theory is based on the recognition of the importance of that component of the wind stress which is in phase with the wave slope rather than the wave height, and this component is intimately connected with the change of phase of an eigenfunction ϕ satisfying the so-called inviscid form of the Orr-Sommerfeld equation (see Section 8) as the singular point of that equation is crossed. Other basic assumptions are:

(a) The air is assumed to be incompressible and to have a mean velocity $U(y)$, with y measured vertically upwards and with x measured in the direction of wind. Viscosity and turbulence, though essential in the maintenance of the wind profile, are assumed to have no other effects. The deviation of the air flow from the mean and the associated pressure perturbation, which arise from the waviness of the water surface, are assumed to be so small that the linearization procedure can be used.

(b) The water is assumed to be incompressible and inviscid, and its motion irrotational. The slope of the waves is assumed small compared with unity. The mean motion of water is entirely neglected.

Before entering into details of Miles theory, it is useful to sketch out its general features. The waviness of the water surface is assumed to take the form

$$\eta(x, t) = ae^{ik(x-ct)}, \tag{74}$$

in which η is the displacement of the water surface from its mean position, a is its amplitude, k is the wave number $2\pi/\lambda$, with λ denoting the wavelength, and c is the complex wave velocity $c_r + ic_i$. The aerodynamic pressure p_a is assumed to be

$$p_a = (\alpha + i\beta)\rho' U_1^2 k\eta, \tag{75}$$

in which ρ' is the density of air and U_1 is a reference speed, which Miles specified to be proportional to the shear velocity $\sqrt{\tau_0/\rho}$ at the water surface. The number α gives the component of the aerodynamic pressure in phase (or, more exactly, in antiphase) with the wave height, and is responsible for wave generation in the Helmholtz model. The number β gives the component of p_a in phase with the wave slope $\partial\eta/\partial x$, and is responsible for wave generation in the Miles model. It turns out that long before the Helmholtz model is capable of wave generation the Miles model is already operative. The number β can be determined from the inviscid form of the Orr-Sommerfeld equation, and once it is determined, c_i can be determined from the equation governing the motion of water.

The equation governing the motion of water is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{76}$$

in which ϕ is the velocity potential. The boundary condition at infinity ($y = -\infty$) is $\phi = 0$, and at the water surface, with ρ denoting the density of water,

$$\eta = -\frac{1}{g}\frac{\partial\phi}{\partial t} - \frac{p_a}{g\rho} \qquad \text{(Bernoulli's equation)}, \tag{77}$$

and, if $y = y_0$ at the undisturbed water surface,

$$\frac{\partial\eta}{\partial t} = \left.\frac{\partial\phi}{\partial y}\right|_{y=y_0} \qquad \text{(kinematic boundary condition)}. \tag{78}$$

The proper solution of (76) is

$$\phi = -ica\exp[ik(x - ct) + k(y - y_0)],$$

which satisfies (76) and (78). Equation (77) demands

$$\eta = \frac{kc^2}{g}ae^{ik(x-ct)} - \frac{p_a}{g\rho} = \frac{kc^2}{g}\eta - \frac{p_a}{g\rho}$$

or, from (75),

$$c^2 = c_w^2 + s(\alpha + i\beta)U_1^2, \tag{79}$$

in which

$$c_w^2 = \frac{g}{k} \qquad \text{and} \qquad s = \frac{\rho'}{\rho}.$$

Equation (79) can be written as

$$c = c_w[1 + \tfrac{1}{2}s(\alpha + i\beta)(U_1/c_w)^2]^{1/2}. \tag{79a}$$

If β is known, the imaginary part of c is known, since s is small. Miles uses c_w as the zeroth approximation for c to find β from the differential equation governing the air motion. The number α is not needed to find β or c_i, only modifies c_r somewhat, and is thus unimportant.

The equations governing the motion of the air are, after linearization,

$$\rho'(u_t + Uu_x + vU_y) = -p_x, \tag{80a}$$

$$\rho'(v_t + Uv_x) = -p_y, \tag{80b}$$

$$u_x + v_y = 0, \tag{80c}$$

in which subscripts indicate partial differentiation, u and v are the perturbation velocity components in the directions of increasing x and z, respectively, and p is the pressure. The equation of continuity permits the use of the stream function ψ, in terms of which

$$u = \psi_y, \qquad v = -\psi_x.$$

If ψ and p are assumed to have the same dependance on x and t as η,

$$\rho'[(c - U)\psi_y + U_y\psi] = p, \tag{81a}$$

$$\rho'k^2(c - U)\psi = p_y, \tag{81b}$$

elimination of p from which produces

$$(U - c)\psi_{yy} - [k^2(U - c) + U_{yy}]\psi = 0. \tag{82}$$

This is the equation used by Rayleigh to study the stability of inviscid fluids in parallel flow. It is the inviscid form of the Orr-Sommerfeld equation (see Section 8). At the value of y for which $U = c$, (82) has a regular singularity.

With the substitutions

$$\xi = ky, \qquad U - c = U_1\, w(\xi), \qquad \text{and} \qquad \psi = -U_1\, \phi(\xi)\, \eta(x, t), \tag{83a, b, c}$$

(82) becomes

$$\phi'' - \left(1 + \frac{w''}{w}\right)\phi = 0. \tag{84}$$

The boundary conditions are

(i) $$\phi \to 0 \quad \text{as} \quad \xi \to \infty.$$

(ii) $$v = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} \quad \text{at} \quad y = y_0 + \eta, \tag{85}$$

or

$$-\frac{\psi_x}{U - c} = ik\eta \quad \text{at} \quad y = y_0 + \eta.$$

If (ii) is to be applied at $y = y_0$ or $\xi = \xi_0$, it is simply

$$\phi_0 = w_0, \quad \text{with} \quad \phi_0 = \phi(\xi_0) \quad \text{and} \quad w_0 = w(\xi_0). \tag{86}$$

Application of the interfacial condition (ii) at $y = y_0$ is not easily justifiable *a priori*, because the velocity gradient of the air at $y = y_0$ may be quite large. However, Miles [1959a] showed that application of (ii) at the interface makes no difference in the result. In this regard, see the use of "wavy" coordinates by Benjamin [1959]. Equation (81a) shows that the perturbation pressure is

$$p = \rho' U_1^2 k(w\phi' - w'\phi)\eta,$$

so that

$$\alpha + i\beta = w_0(\phi_0' - w_0').$$

Since $w(\xi)$ is real in the zeroth approximation,

$$\beta = \text{Im}\,(w_0\phi_0'). \tag{87}$$

In the first approximation, β can be evaluated by multiplying (84) by ϕ^* (the complex conjugate of ϕ) and integrating, by parts if necessary, between ξ_0 and ∞. The result is

$$\int_{\xi_0}^{\infty} \left\{|\phi'|^2 + \left(1 + \frac{w''}{w}\right)|\phi|^2\right\} d\xi = [\phi^*\phi']_{\xi_0}^{\infty} = -w_0\phi_0'. \tag{88}$$

Thus

$$\beta = \text{Im}\,(w_0\phi_0') = -\text{Im} \int_{\xi_0}^{\infty} \frac{w''}{w}\,|\phi^2|\,d\xi. \tag{89}$$

The imaginary part of this integral arises from the crossing of the point $y = y_c$, at which $U = c$ and hence $w = 0$. It depends crucially on whether the path of integration is below or above the singular point. If c_i is slightly positive, there is no difficulty, for the path is then below the singular point $y = y_c$ (complex). If c_i is zero, as it is in the zeroth approximation, an indentation around it is necessary, and Lin [1955, Section 4.3, and Chapter 8] has shown that the half circle of the indentation should be below the singular point (Fig. 46). Thus

$$\beta = -\pi|\phi_c|^2 \frac{w_c''}{w_c'}.$$

The evaluation of β thus depends on the evaluation of ϕ_c. Miles used a logarithmic velocity profile and obtained an approximate form for ϕ_c, and hence an approximate value for β. If w_c'' is negative, as it indeed is for a logarithmic velocity profile, β is positive.

FIGURE 46. The contour for crossing the critical point.

To compute the least wind speed necessary to generate waves Miles computed the rate of damping due to viscous dissipation* in water and equated this rate to c_i computed from β. Miles carefully considered the velocity profile and obtained the result that the minimum speed at two meters above the water surface for the initiation of *gravity* waves in deep water is of the order of 100 cm/sec, with the corresponding c_r or c_w somewhere between 40 and 50 cm/sec, as compared with the 650 cm/sec needed for wave generation by the Helmholtz model.

It does seem rather strange that the entire mechanism of wave generation should depend so critically on the values of ϕ, w, and w'' at $y = y_c$. Mathematically it is clear that these values determine that part of the air pressure which is in phase of the wave slope. The physical explanation of this situation by the concept of the vortex force, and the description of how the air contrives to transmit the happenings at the critical layer to the water have been given in an illuminating paper by Lighthill [1962]. Miles' success in overcoming the many practical difficulties in dealing with turbulent flows, his recognition of the importance of the small part of p_a in phase with wave slope, and his ability to construct a valid theory upon this idea have happily combined to contribute an important result to modern hydrodynamics.

Miles also improved the Helmholtz model by considering the effect not of a vortex sheet but of distributed vorticity (in the wind) on wave generation. He found that the Helmholtz model so improved is not operative for the air-water interface, but does govern the stability of wind blowing over a very viscous oil [1959b].

7. STABILITY OF STRATIFIED FLOWS

The stability of shear flows of a continuously stratified fluid is a fascinating phenomenon of great importance to meteorology. The number of workers in this field has not been large. Before the work of these workers is discussed in

* Remember that the assumed irrotationality of the motion of water does not contradict the consideration of viscous dissipation. A viscous fluid can be in essentially irrotational motion, and irrotational motion is in general associated with deformation and therefore viscous dissipation.

detail, a look at (72) will bring out some of the parameters governing, or affecting, stability, although (72) is based on the restriction to periodicity in the direction of flow. Neglecting T, letting ρ approach ρ', and using h as a reference length, we see that the brace in (72) can be written as

$$\frac{1}{kh}\frac{gh^2}{2\bar{\rho}}\frac{d\bar{\rho}}{dy} - \frac{h^2}{4}\left(\frac{dU}{dy}\right)^2 = \frac{h^2}{4}\left(\frac{dU}{dy}\right)^2\left[\frac{2}{kh}g\beta\left(\frac{dU}{dy}\right)^{-2} - 1\right],$$

in which $\bar{\rho}(y)$ is the density of the undisturbed fluid, kh is the dimensionless wave number, and

$$J = g\beta\left(\frac{dU}{dy}\right)^{-2}$$

is the Richardson number, with

$$\beta = -\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dy}.$$

Since the stability of Helmholtz's model is decided by the sign of the quantity inside the brace in (72), with $T = 0$, we expect the wave number kh and the Richardson number, *among other things*, to affect the stability of the flow. The Richardson number is a measure of the ratio of gravity forces to inertial forces. If it is high, the flow is likely to be stable. If it is low, the flow may be unstable. Of course for a continuously varying $\bar{\rho}$ and U, the Richardson number varies from elevation to elevation.

With $U(y)$ now indicating the velocity of the undisturbed fluid moving steadily and horizontally in the x-direction and $\bar{\rho}(y)$ its density, u and v the x- and y-components of the perturbation velocity, and ρ the perturbation in density, the linearized equations of motion are

$$\bar{\rho}(u_t + Uu_x + vU') = -p_x, \tag{90}$$

$$\bar{\rho}(v_t + Uv_x) = -p_y - g\rho, \tag{91}$$

in which the subscripts denote partial differentiation, U' now stands for dU/dy, and p denotes the deviation from the hydrostatic pressure in the undisturbed fluid. The equation of continuity is

$$u_x + v_y = 0, \tag{92}$$

which allows the use of the stream function ψ, in terms of which

$$u = \psi_y, \qquad v = -\psi_x. \tag{93}$$

The equation of incompressibility is

$$\rho_t + U\rho_x + v\bar{\rho}' = 0, \tag{94}$$

in which

$$\bar{\rho}' = \frac{d\bar{\rho}}{dy}.$$

The stability problem is governed by (90), (91), (92), (94), and the boundary conditions. If the perturbation quantities grow with time, the flow is unstable; otherwise it is stable.

If η stands for the vertical displacement of an isopycnic line from its mean position, then

$$\eta_t + U\eta_x = v = -\psi_x. \tag{95}$$

For periodic disturbances, all perturbation quantities are assumed to have the exponential factor $e^{ik(x-ct)}$, and we shall follow Miles [1961] in using η as the primary dependent variable. Thus,

$$\eta(x, y, t) = F(y)\, e^{ik(x-ct)}, \tag{96}$$

and it follows from (95) and (93) that

$$\psi = -(U - c)\eta, \qquad u = -[(U - c)\eta]', \qquad v = ik(U - c)\eta. \tag{97}$$

Equation (90) then gives

$$p = \bar{\rho}(U - c)^2\eta', \tag{98}$$

and (94) gives

$$\rho = -\bar{\rho}'\eta. \tag{99}$$

Substituting (97), (98), and (99) in (91), we have

$$[\bar{\rho}(U - c)^2F']' + \bar{\rho}[\beta g - k^2(U - c)^2]F = 0, \tag{100}$$

which is more convenient to use than the equation obtained by taking ψ to be the primary dependent variable. At a rigid wall, $F = 0$. Subsequent discussion will be divided into subsections, for the sake of clarity.

7.1. *The Work of Taylor and Goldstein*

The first paper after the work of Helmholtz [1868] and Kelvin [1871] and dealing with the stability of a heterogeneous fluid in shear flow is that of Taylor [1931], which is based on part of the work contained in an essay for which the Adam Prize was awarded to him in 1915.

Case 1. Semi-infinite fluid with $U = \alpha_1 y$, $\bar{\rho} = \rho_0 e^{-\beta y}$, rigid boundary at $y = 0$.

Case 2. A middle layer of thickness h and density ρ_3, and with $U = U_1 + \alpha y$, y being measured from its lower boundary, a semi-infinite lower layer of density ρ_1 and velocity U_1, and a semi-infinite upper layer of density ρ_2 and velocity U_2. The three layers are contiguous.

Case 3. Same as Case 2, except that $U = \alpha_1 y$ throughout.

Case 4. Four superposed layers with $U = \alpha_1 y$ throughout, and two finite middle layers of equal depth.

Goldstein [1931], whose paper was published together with Taylor's in the *Proceedings of the Royal Society*, considered the following cases:

Case 1. Essentially the same as Taylor's Case 2. The density jumps at the two interfaces are of the same magnitude exactly. The origin of y is taken at the midpoint of the middle layer, so that U is an odd function of y.

Case 2. Same as his Case 1, except that the middle layer in Case 1 is split into three. The four density jumps are equal and the velocity varies linearly with y in the three middle layers.

Case 3. $U = 0$. A middle layer in which the density distribution is exponential, semi-infinite upper and lower layers, each of constant density. The density is continuous and statically stable throughout. (This is of course a problem of wave progression only, because the configuration is always stable.)

Case 4. The density distribution is the same as in Case 3, but the velocity is the same as in Case 1.

Taylor's Case 2 and Goldstein's Case 1 are the same, and their results agree. Since the vorticity of the primary flow in each of the three layers is constant (although nonzero in the middle layer), the equation governing the *perturbation* flow in each layer is the Laplace equation, and the solution by separation of variables is straightforward. The stability diagram as given by Goldstein is shown in Fig. 47. It can be compared with the stability diagram of the Helmholtz model, obtained by setting the brace in (72), with $T = 0$, equal to zero. The graph is a straight line described by

$$k = 2g \frac{\Delta\rho}{\rho} (\Delta U)^{-2},$$

and is shown in Fig. 48. To the left of the line the flow is stable and to the right unstable. In Fig. 47 the region between the two curves is a band of instability, and the rest of the plane is a region of stability. It appears that the distribution of vorticity over a finite region has the effect of stabilizing the short waves and leaves only waves of intermediate wavelengths unstable.

For Goldstein's Case 2 he found two bands instead of one, similar to the band shown in Fig. 47. His Case 3 needs no discussion, since the problem is not one of stability. His Case 4 presents an anomaly which will be taken up later.

Taylor pointed out that in his Case 2 (Goldstein's Case 1) the unstable range of the relative velocity ΔU (or $U_2 - U_1$) is a very narrow one near that for which the backward moving free wave on the upper interface moves with the same speed as the forward moving free wave at the lower interface, so that the instability might be regarded as due to a sort of resonance of two waves with the same wavelength moving in space with the same speed. The same is found in his Case 3. In his Cases 3 and 4 and for very long waves Taylor found that the critical Richardson number $g\ \Delta\rho/\rho h\alpha_1^2$ is 2 and 2.11, respectively, $\Delta\rho$ being the *small* increment of ρ from layer to layer, and h being the depth of the middle layer in Case 3, and the depth of each of the middle layers in Case 4.

Taylor's Case 1 and Goldstein's Case 4 needs fuller discussion. Taylor found for his Case 1 that when the Richardson number J (constant in this case) is greater than $\frac{1}{4}$, there is neutral stability and progressive waves exist, whereas for $0 < J < \frac{1}{4}$ harmonic waves, whether stable or unstable, cannot exist. This immediately raises the question of what the fluid will do when J falls within that range. First, it should be realized that this puzzling situation is far from unique. A homogeneous fluid of constant density and zero viscosity and con-

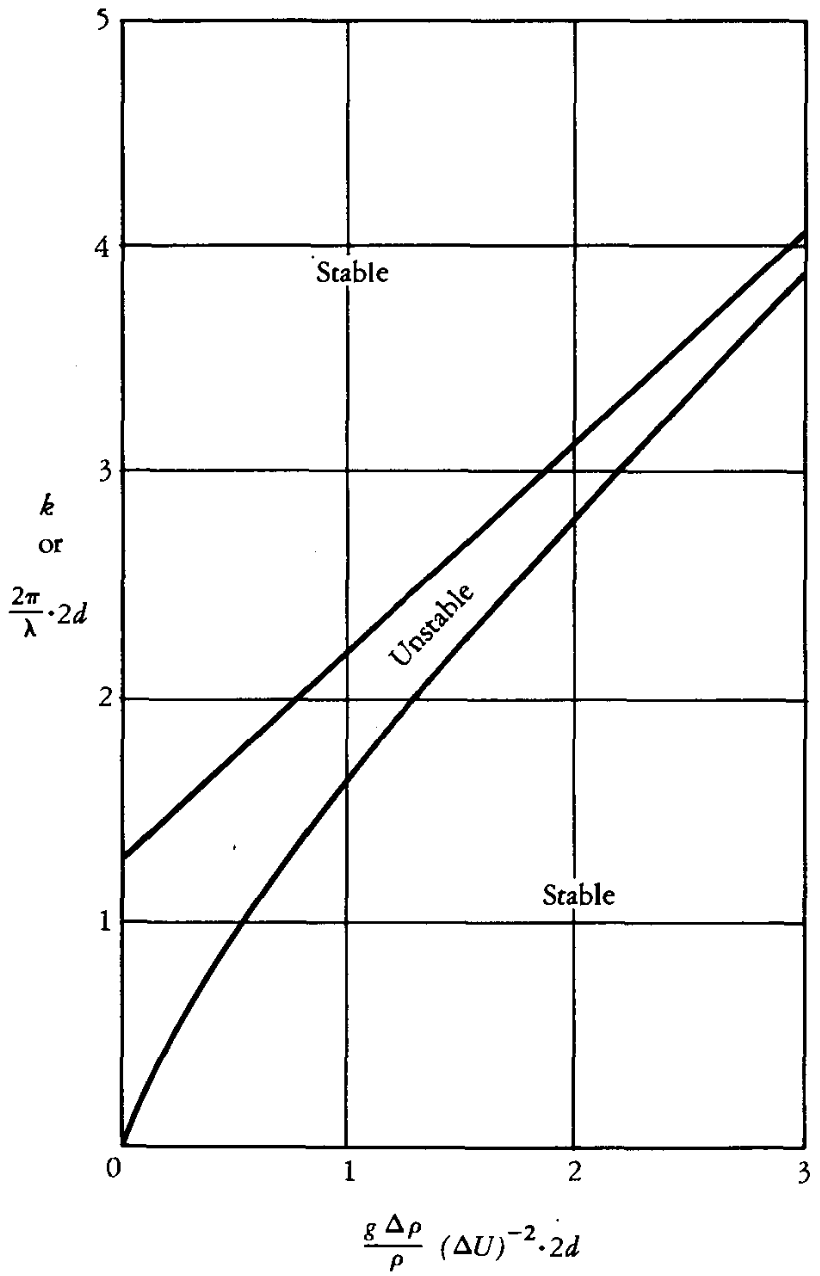


FIGURE 47. Stability chart for Goldstein's Case 1. The density in the upper semi-infinite layer is $\rho - \Delta\rho/2$, that in the middle layer is ρ, and that in the lower semi-infinite layer is $\rho + \Delta\rho/2$. The velocity is U in the upper layer, $-U$ in the lower layer, and varies linearly in the middle layer of depth $2d$.

fined between two rigid horizontal boundaries exhibits the same behavior. If the variation with x is sinusoidal, the vertical velocity cannot be zero at both boundaries. A slightly less obvious case is the case of plane Couette flow of an inviscid fluid of constant density. In such cases, as in Taylor's Case 1, the question to be answered is: Given an initial disturbance, what will its subsequent development be? For both of the simple cases mentioned, the equation governing the vorticity ζ of a two-dimensional *perturbation* flow (which is the entire vorticity in the case of an ideal fluid at rest or in uniform flow) is

$$\frac{D\zeta}{Dt} = 0,$$

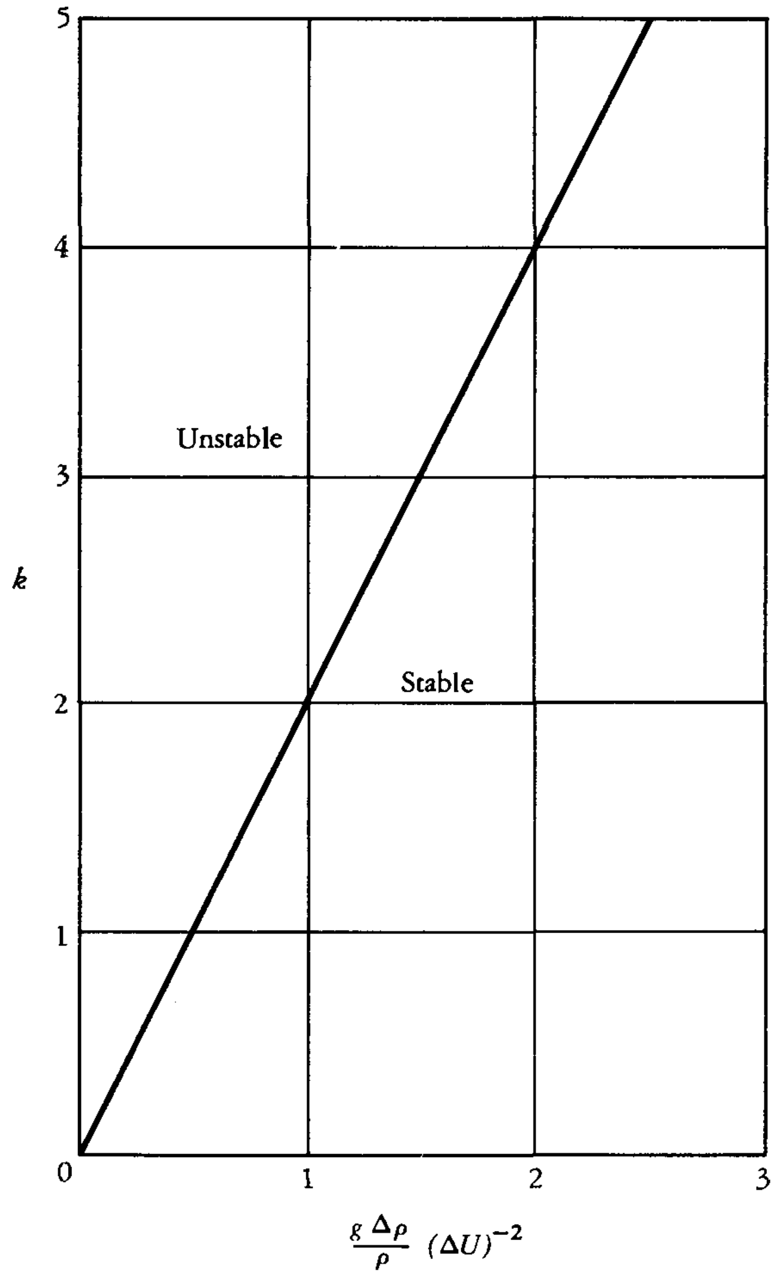


FIGURE 48. The stability diagram for the Helmholtz model. Both coordinates are in units of 1/(unit length).

which was due to Helmholtz. One may, as K. M. Case [1960a] did for plane Couette flow and as Orr [1907a, b] did long before him, use Laplace transform to trace out the subsequent motion. But the main feature is given by the conservation of ζ. If the maximum of $|\zeta|$ in the entire field at $t = 0$ is a, subsequent $|\zeta|$ will not be anywhere larger than a, and the fluid cannot be considered as unstable, whether or not the initial disturbance is infinitesimal.* For Taylor's Case 1, Eliassen *et al.* [1953] and K. M. Case [1960b] solved the initial-value problem by the use of the Laplace transform, and Dyson [1960] showed that, if the exponential factor $e^{ik(x-ct)}$ is used, there are infinitely many discrete real eigenvalues for c for each value of k if $J > \frac{1}{4}$, either one or no such eigenvalue if $0 < J < \frac{1}{4}$, and in no case is there any complex eigenvalue for c if J is positive. Aside from these discrete modes, Case's initial-value analysis [1960b] shows that the remaining part of the initial disturbance dies out as $t \to \infty$ as $1/\sqrt{t}$ if $J > \frac{1}{4}$ and as t^n, with $n = (\frac{1}{4} - J)^{1/2} - \frac{1}{2}$, if $J < \frac{1}{4}$. Thus all the results obtained for Taylor's Case 1 indicate that the flow is stable if the stratification is statically stable ($J > 0$).

For his Case 4, Goldstein obtained the result that harmonic oscillations are stable for $J > \frac{1}{4}$, and unstable for all wavelengths for $0 < J < \frac{1}{4}$. Miles [1960] pointed out that, as the thickness $2h$ of the middle layer becomes smaller and smaller, Goldstein's Case 4 should approach the Helmholtz model. If the difference between the densities at the two interfaces is denoted by $\Delta\rho$ and the difference in velocity at the two places is denoted by ΔU, (72) for the Helmholtz model predicts stability (for the limiting case) for

$$k < 2g\,\frac{\Delta\rho}{\rho}\,(\Delta U)^{-2},$$

or

$$\alpha = kh < g\left(\frac{\Delta\rho}{2h\rho_0}\right)\left(\frac{2h}{\Delta U}\right)^2 = J, \tag{101}$$

in contradiction to Goldstein's conclusion. The situation has now been clarified by Miles and Howard [1964, 331]. As will be seen, the results of Drazin and Howard [1961] also contradict Goldstein's result for his Case 4.

7.2. *General Results*

Shortly after the papers of Taylor and Goldstein were published, there appeared an important paper of John L. Synge [1933] in a Canadian journal, which for a good many years escaped the notice of many later workers

* Thus the linearization procedure adopted by Orr and Case is not necessary to reach the conclusion of stability, if the fluid is *inviscid and homogeneous*. If the vorticity of the main flow is not constant, the same argument shows that the total vorticity maintains its maximum value throughout the subsequent motion. However, there can be transfer of vorticity between the main flow and the perturbation flow, and hence instability may occur within the frame work of the linear approach.

(including the present author), who have rediscovered some of Synge's results. In contrast with the results of Taylor and Goldstein, which are for special velocity and density distributions, Synge's are general in nature.

The analyses of Eliassen, Høiland, and Riis [1953] and of Case [1960b] for Taylor's Case 1 (for finite depth in the paper of Eliassen *et al.* and for infinite depth in the paper of Case) indicate that dynamic instability of a statically stable fluid in shear flow cannot be other than exponential. While this result has been demonstrated only for Taylor's Case 1, it is probably valid in general. Thus (100), or an equivalent one in terms of the stream function, can be used as the basis of analysis. The main results concern (a) sufficient conditions for stability, (b) the range of c_r and c_i for unstable flows, (c) the Reynolds stress, and (d) the anomalous character of the singular neutral modes as regards their infinite kinetic and potential energies. Items (a) and (b) are intimately related.

7.2.1. *Synge's Results.* If the stream function ψ is assumed to be

$$\psi = f(y)\, e^{ik(x-ct)} \tag{102}$$

then since, on the one hand,

$$v = -\psi_x = -ik\, f(y)\, e^{ik(x-ct)},$$

and, on the other, (97) gives

$$v = ik(U - c)\eta = ik(U - c)F(y)\, e^{ik(x-ct)},$$

it follows that

$$f(y) = -(U - c)F(y). \tag{103}$$

The equation satisfied by $f(y)$ can be obtained either directly from the Euler equations or from (100), and is

$$(\bar{\rho} f')' + \left[\frac{(\bar{\rho}U')'}{c - U} - k^2\bar{\rho} - \frac{g\bar{\rho}'}{(c - U)^2}\right] f = 0. \tag{104}$$

The boundary conditions are, if rigid boundaries are situated at $y = 0$ and $y = d$,

$$f(0) = 0 \qquad \text{and} \qquad f(d) = 0. \tag{105}$$

The complex conjugates of (104) and (105) are of course valid. Now if (104) is multiplied by f^* (the complex conjugate of f) and its complex conjugate by f, the difference taken, and the results are integrated, we have

$$\int_0^d [(\bar{\rho} f')' f^* - (\bar{\rho} f^{*\prime}) f]\, dy = \int_0^d \left[(\bar{\rho}U')'\left(\frac{1}{U - c} - \frac{1}{U - c^*}\right) + g\bar{\rho}'\left(\frac{1}{(U - c)^2} - \frac{1}{(U - c^*)^2}\right)\right] f f^*\, dy.$$

The left-hand side can be integrated by parts, and the boundary conditions used. The result is zero. After some simplifications this equation becomes

$$0 = c_i \int_0^d [(U - c_r)^2 + c_i^2]^{-2} \{(\bar{\rho}U')'[(U - c_r)^2 + c_i^2)] + 2(U - c_r)g\bar{\rho}'\} ff^* \, dy. \tag{106}$$

Thus if $c_i \neq 0$, the quantity in the brace must change sign in the interval $0 < y < d$, and hence must be zero for some value of y. This result was originally proved by Synge, and independently by Yih [1957, as implied in equation (51) of that paper, in which the factor of 2 is erroneously missing] and by Drazin [1958]. If $(\bar{\rho}U')'$ is of the same sign throughout, (106) gives

$$\max \left| \frac{g\bar{\rho}'}{(\bar{\rho}U')'} \right| > |c_i|, \tag{107}$$

which again was proved by Synge, and later independently by Yih [1957], except for an erroneous factor of 2. For a homogeneous fluid, $\bar{\rho}' = 0$, and (106) produces the result that if $c_i \neq 0$ then U'' must change sign in the interval. This is the well known theorem of Rayleigh.

7.2.2. *Miles' Results*. Miles [1960] distinguished the neutral modes which are contiguous with unstable modes from neutral modes which do not have that property, and called the collection of the former kind the stability boundary. That a neutral mode may not be on the stability boundary is obvious. Take, for instance, gravity waves in a homogeneous liquid with a free surface. The wave motion is always a neutral mode, but the fluid is certainly not on the verge of becoming unstable. Another example is furnished by the Helmholtz model. There are infinitely many wave numbers for which the flow is stable. Only one is on the "stability boundary." It turns out that this distinction is rather significant.

Miles derived from (104) by the Frobenius method of power expansion and on the assumption of analyticity of U and $\bar{\rho}$ that, for singular neutral modes for which $U = c$ at $y = y_c$,

$$\sqrt{\bar{\rho}} f(y) = (y - y_c)^{(1/2)(1 \pm \nu)} w_{\pm}(y), \tag{108}$$

in which $w_{\pm}$ are analytic functions of y near $y = y_c$, and

$$\nu = (1 - 4J_c)^{1/2},$$

J_c being the Richardson number J evaluated at $y = y_c$. Since $u \sim f'(y)$, and

$$\eta = \frac{f(\eta)}{c - U} e^{ik(x-ct)}, \tag{109}$$

the solution of $f(\eta)$ associated with $-\nu$ indicates that both the kinetic energy T and potential energy P are infinite for singular neutral modes, with

$$T = \frac{1}{2}\int_0^d \bar{\rho}(\overline{u^2 + v^2})\,dy, \qquad P = \frac{1}{2}g\int_0^d \beta\bar{\rho}\overline{\eta^2}\,dy.$$

(The bars on $u^2 + v^2$ and on η^2 indicate the average over one wavelength.) This is a troublesome situation which is physically inadmissible.* Possibly the consideration of diffusive effects will remove this difficulty.

Since η must be zero at $y = 0$ and $y = d$, (108) and (109) show that

$$c \neq U(0) \qquad \text{and} \qquad c \neq U(d), \tag{110}$$

provided that U is analytic, so that $c - U$ cannot vanish as a fractional power of $y - y_c$. In fact, multiplication of (100) by F^* and integration, by parts, if necessary, produces the result

$$g\int_0^d \bar{\rho}\beta|F|^2\,dy = \int_0^d \bar{\rho}(U - c)^2(|F'|^2 + k^2|F|^2)\,dy, \tag{111}$$

provided $c_i \neq 0$. The imaginary part of this is

$$2c_i\int_0^d (U - c_r)(|F'|^2 + k^2|F|^2)\,dy = 0, \tag{111a}$$

which means for unstable modes ($c_i \neq 0$)

$$U_{\min} < c_r < U_{\max}. \tag{112}$$

If the velocity U is monotonically increasing with y, this implies that

$$U(0) < c_r < U(d).$$

These results are contained in the richer semicircle theorem of Howard, which will be presented later. Equations (110) and (112) also imply that a stability boundary consists of singular neutral modes, for which $c_i = 0$ and $U = c$ in $0 < y < d$. (The reverse, that singular neutral modes lie on a stability boundary, is also true. See Miles [1963].) Equation (111) is valid so long as $U \neq c$ in $0 < y < d$. Hence it also implies that a nonsingular neutral mode cannot exist if $\beta < 0$—that is, if the fluid is statically unstable.

Another interesting result of Miles concerns the Reynolds stress

$$\tau = -\bar{\rho}\overline{uv}.$$

Expressing u and v in terms of f(6), we have

$$\tau = \tfrac{1}{2}k\bar{\rho}(f'f^*)_i e^{2kc_it}, \tag{113}$$

in which the subscript i denotes the imaginary part. Differentiating (113) and utilizing (104) to evaluate $(\bar{\rho}f')'$ yields

* However, Dr. Louis Howard pointed out to the author that the situation is not so bad if one considers the initial-value problem. The singular neutral modes are then isolated modes imbedded in modes with a continuous spectrum in k, and their infinite energies do not imply infinite energies of the whole disturbance.

$$\frac{\partial \tau}{\partial y} = -k^{-1} h_i \bar{\rho} \overline{v^2}, \tag{114}$$

in which

$$h_i = c_i \left[\frac{2\beta g(U - c_r)}{|U - c|^4} - \frac{(\bar{\rho} U')'}{\bar{\rho} |U - c|^2} \right].$$

Thus, if $c_i = 0$, $\partial\tau/\partial y$ is everywhere zero except possibly at a singular point. If there is no singular point or if there is only one such point (which is the case if U is monotonic), the Reynolds stress is everywhere zero, because it is zero on both boundaries.

We shall now present the main theorem in Miles' work, which states that if $g\beta \geq \frac{1}{4}U'^2$ everywhere the flow is stable. We shall give Howard's proof of Miles' theorem, for the sheer delight of its simplicity. Setting $W = U - c$ and $G = W^{1/2}F$, Howard [1961] obtained

$$(\bar{\rho} W G')' - [\tfrac{1}{2}(\bar{\rho} U')' + k^2 \bar{\rho} W + \bar{\rho} W^{-1}(\tfrac{1}{4}U'^2 - g\beta)]G = 0. \tag{115}$$

The boundary conditions for G are $G(0) = 0 = G(d)$. Multiplication of (115) by G^* and integration (by parts if necessary) yields (with limits omitted)

$$\int [\bar{\rho} W(|G'|^2 + k^2|G|^2) + \tfrac{1}{2}(\bar{\rho} U')'|G|^2 + \bar{\rho}(\tfrac{1}{4}U'^2 - g\beta) W^* |G/W|^2] \, dy = 0. \tag{116}$$

If $c_i \neq 0$, the imaginary part of (116) gives

$$\int \bar{\rho}(|G'|^2 + k^2|G|^2) \, dy + \int \bar{\rho}(g\beta - \tfrac{1}{4}U'^2)|G/W|^2 \, dy = 0, \tag{117}$$

which is clearly impossible if $g\beta \geq \frac{1}{4}U'^2$ everywhere. Hence if $g\beta \geq \frac{1}{4}U'^2$ everywhere the flow is stable. This is Miles' theorem. If U' is nowhere zero, Miles' sufficient condition can be written as $J > \frac{1}{4}$. (Note: If c is an eigenvalue, c^* also is one. Hence $c_i \neq 0$ implies instability.) Howard's proof also shows that Miles' assumption of analyticity of U and $\bar{\rho}$ is not necessary.

7.2.3. *Howard's Semicircle Theorem.* For $c_i \neq 0$, the real part of (111) is

$$\int_0^d \bar{\rho}[(U - c_r)^2 - c_i^2](|F'|^2 + k^2|F|^2) \, dy - \int_0^d g\bar{\rho}\beta|F|^2 \, dy = 0. \tag{111b}$$

Howard [1961] took

$$Q = \bar{\rho}(|F'|^2 + k^2|F|^2),$$

and, since $c_i \neq 0$, wrote (111a) and (111b) as

$$\int UQ \, dy = c_r \int Q \, dy, \tag{118}$$

$$\int U^2 Q \, dy = (c_r^2 + c_i^2) \int Q \, dy + \int g\bar{\rho}\beta|F|^2 dy = 0. \tag{119}$$

In obtaining (119) from (111b), (118) has been used. Let a be the minimum and b the maximum of $U(y)$, so that

$$a \leq U(y) \leq b.$$

Then

$$\begin{aligned} 0 \geq \int (U - a)(U - b)Q\,dy &= \int [U^2Q - (a + b)UQ + abQ]\,dy \\ &= [c_r^2 + c_i^2 - (a + b)c_r + ab]\int Q\,dy + \int g\bar{\rho}\beta|F|^2\,dy \\ &= \{[c_r - \tfrac{1}{2}(a + b)]^2 + c_i^2 - [\tfrac{1}{2}(a - b)]^2\}\int Q\,dy + \int g\bar{\rho}\beta|F|^2\,dy. \end{aligned}$$

Since $\beta > 0$ for the statically stable fluid under consideration, and $Q > 0$ by definition,

$$[c_r - \tfrac{1}{2}(a + b)]^2 + c_i^2 \leq [\tfrac{1}{2}(a - b)]^2. \tag{120}$$

Hence Howard's semicircle theorem: The complex wave velocity c for any unstable mode must lie inside the semicircle in the upper half-plane which has the range of U for diameter. This is a very beautiful theorem in the theory of hydrodynamic stability.

From (117) and noting that $|W|^{-2} \leq c_i^{-2}$. Howard also obtained

$$\begin{aligned} k^2\int \bar{\rho}|G|^2\,dy &= \int \bar{\rho}(\tfrac{1}{4}U'^2 - g\beta)|G/W|^2\,dy - \int \bar{\rho}|G'|^2\,dy \\ &\leq \frac{1}{c_i^2}\max(\tfrac{1}{4}U'^2 - g\beta)\int \bar{\rho}|G|^2\,dy, \end{aligned}$$

so that

$$k^2c_i^2 \leq \max(\tfrac{1}{4}U'^2 - g\beta), \tag{121}$$

giving an upper bound for $k^2c_i^2$, kc_i being the rate of growth of the disturbance.

7.3. *The Drazin-Howard Theory*

For the stability of a slightly stratified (small J) fluid in shear flow in infinite domain against the formation of long waves (small k), Drazin and Howard [1961] used a series expansion in k^2 and J. The velocity profiles considered are either odd or even, and $\bar{\rho}(y) - \bar{\rho}(0)$ is considered to be an odd function of y. Solutions are given for positive y and negative y separately, and the results are matched at $y = 0$. For details the reader is referred to the original paper. Only the final result is given here:

$$\begin{aligned} \alpha(W_\infty^2 + W_{-\infty}^2) - 2J + \int_{-\infty}^{\infty} &[\alpha(W^2 - W_\infty^2) + J(1 - \lambda)] \\ &\times [\alpha(W^2 - W_{-\infty}^2) + J(1 - \lambda)]\frac{dy}{W^2} + \cdots = 0, \end{aligned} \tag{122}$$

in which $W = U - c$, all referred to a reference velocity V, and α is the dimensionless wave number. For *small* change in $\bar{\rho}$,

$$\lambda(y) = \frac{\bar{\rho}(-\infty) + \bar{\rho}(+\infty) - 2\bar{\rho}(y)}{\bar{\rho}(-\infty) - \bar{\rho}(+\infty)},$$

and

$$J = g\frac{L}{V^2}\frac{\bar{\rho}(-\infty) - \bar{\rho}(+\infty)}{\bar{\rho}(-\infty) + \bar{\rho}(+\infty)},$$

L being a reference length. For

$$U = \lambda = \begin{cases} -1 & \text{for } \ y \leq -1, \\ \ \ y & \text{for } \ -1 \leq y \leq 1, \\ \ \ 1 & \text{for } \ y \geq 1, \end{cases} \tag{123}$$

the authors obtained (for small α)

$$J = \alpha - \tfrac{2}{3}\alpha^2 \tag{124}$$

as the stability boundary, in agreement with (101). For small β, Goldstein's Case 4 is essentially the same as the case described by (123), but his result is not in agreement with (124).

Note that (122) gives the eigenvalues of c in terms of α, λ, J, and U, and that it is not necessary to know the eigenfunction denoted by ϕ in the paper of Drazin and Howard. Herein lies its usefulness.

7.4. *Yih's Sufficient Conditions for Stability or Instability*

Miles' theorem and Howard's semicircle theorem can be generalized to include density discontinuities [Yih, 1970a]. The generalization is straightforward and will not be repeated here.

Miles' criterion for stability ($J \geq \frac{1}{4}$) is not the natural generalization of Rayleigh's well-known sufficient condition for the stability of a homogeneous fluid in shear flow. This generalization was given by Yih [1970a], who also gave sufficient conditions for instability. His results will now be given.

Consider a parallel flow between two rigid horizontal boundaries at a distance d apart, with velocity given by $U(y)$, y retaining its meaning that it has heretofore in this section. The equation governing stability or instability is (104), which can be made dimensionless by the use of the new variables

$$\hat{f} = \frac{f}{Vd}, \qquad \hat{\rho} = \frac{\bar{\rho}}{\rho_0}, \qquad \hat{y} = \frac{y}{d}, \qquad \hat{U} = \frac{U}{V}, \qquad \hat{c} = \frac{c}{V}, \tag{125}$$

where ρ_0 is a reference density and V a reference velocity. Then (104) becomes, after the carets are dropped,

$$(\bar{\rho}f')' + \left[\frac{(\bar{\rho}U')'}{c - U} - \alpha^2\bar{\rho} - \frac{N\bar{\rho}'}{(c - U)^2}\right]f = 0, \tag{126}$$

in which everything is now dimensionless, the primes indicate differentiation with respect to the dimensionless y,

$$\alpha = kd \tag{127}$$

is the dimensionless wave number, and

$$N = gd/V^2 \tag{128}$$

is actually the reciprocal of the square of a Froude number. The appearance of N does not necessarily signify the importance of surface waves, since it appears even if the upper boundary is fixed. The fact that it is associated by multiplication to $\bar{\rho}'$ indicates that the entire term represents the effect of gravity in a stratified fluid in shear flow.

Henceforth in this paper we shall consider rigid boundaries only, for which the boundary conditions are

$$f(0) = 0 \qquad \text{and} \qquad f(1) = 0, \tag{129}$$

to be imposed on the function f in (126).

It is then clear that the system consisting of (126) and (129) gives, for a nontrivial solution, a relationship

$$F_1(\alpha, N, c) = 0. \tag{130}$$

Since c is complex, (130) has a real part and an imaginary part. When c_i is set to zero and c_r eliminated from the two component equations, a relationship

$$F_2(\alpha, N) = 0, \tag{131}$$

if one such exists, gives the neutral-stability curve. It is possible, however, that c is real for all values of α and N, in which case $c_i = 0$ in the entire N-α plane, and then, of course, there is no neutral-stability curve, because one component equation of (130) is $c_i = 0$ and the other is simply (130) itself, with the c therein real.

In this section, we shall assume $\bar{\rho}$ and U to be continuous, analytic, and monotonic. Furthermore, we assume $\bar{\rho}' < 0$ throughout. We now recall the following known results:

(i) Miles' criterion;
(ii) Howard's semicircle theorem;
(iii) if an eigenfunction exists for (c_0, α_0, N_0), then near that point c is a continuous function of α and N [Miles, 1963; Lin, 1945].

7.4.1. *Sufficient Conditions for Stability.* Under the assumptions we have made on $\bar{\rho}$ and U, and in view of the known results just cited, we conclude that the nonexistence of any singular neutral mode, which is a mode with a real c equal to U at some point in the flow, implies the nonexistence of unstable modes. The reason is as follows. In the N-α plane there is always a region of stability. For we can imagine g and hence $J(y)$ to increase indefinitely, until $J(y)$ is everywhere greater than $\frac{1}{4}$, which is attainable since β is nowhere zero. Thus there is a region of large N for which the flow is stable. If unstable modes exist there must then be a stability boundary dividing the region of stability from the region of instability, and hence a neutral-stability curve. As we approach that curve from the region of instability, c_r being within the range of U so long as $c_i \neq 0$ and continuous in α and N so long as c is an eigenvalue, according to (iii) above, in the limit when $c_i = 0$, c_r must be within the range of U, i.e., the limiting mode must be a singular neutral mode. Hence the nonexistence of a singular neutral mode implies the nonexistence of unstable modes. In fact even the existence of special singular neutral modes for which c equals the maximum or minimum of U would not imply the existence of contiguous unstable modes, as a consequence of the semi-circle theorem of Howard. Hence we need not be concerned with these special border cases. In demonstrating the nonexistence of unstable moses it is sufficient to demonstrate the nonexistence of singular neutral modes with $a < c < b$, where a is the minimum and b the maximum of U.

Miles [1961, p. 507] has shown that singular neutral modes are impossible for monotonic U if $J(y) > \frac{1}{4}$ everywhere. In his demonstration he actually showed that a singular neutral mode with a $J(y_c) > \frac{1}{4}$ at the place $y = y_c$ where $U = c$ is impossible. Hence we need only consider the case $J(y_c) \leq \frac{1}{4}$ in our search for the nonexistence of singular neutral modes. For $J_c \equiv J(y_c) = \frac{1}{4}$, one solution of (126) is

$$f_1 = (y - y_c)^{1/2} w_1, \tag{132}$$

where

$$w_1 = 1 + A(y - y_c) + \cdots, \tag{133}$$

with

$$A = \left[(1 + J)\frac{(\bar{\rho}U')'}{\bar{\rho}U'} - \frac{J\bar{\rho}''}{\bar{\rho}'} + \frac{1}{2}(\ln \bar{\rho})'\right]_c, \tag{134}$$

provided U' does not vanish at $y = y_c$. (We shall consider monotonic U only. Hence this restriction on U' does not affect our results.) The other solution is found by assuming it to be of the form $f_1 h$, substituting it into (126), and solving for h. The result, after division by a constant (which is $\bar{\rho}_c$ or $\bar{\rho}$ at y_c), is

$$f_2 = f_1 \ln (y - y_c) - [2A + (\ln \bar{\rho})'_c](y - y_c)^{3/2}[1 + B(y - y_c) + \cdots]. \tag{135}$$

Now the Reynolds stress, defined by

$$\tau = -\overline{\bar{\rho}uv}, \tag{136}$$

where the bar over uv means time or space average, can be expressed in terms of f as

$$\tau = \frac{\rho_0 V^2}{2} \alpha(f'f^*)_i e^{2\alpha c_i t}, \tag{137}$$

in which the asterisk denotes the complex conjugate, and t, now in terms of d/V, is dimensionless, as is f. Considering the singular neutral case, for which $c_i = 0$, it is easy to see from (132) and (135) that if $(f'f^*)_i$ is zero for $y > y_c$, it is equal to $-i\pi$ for $y < y_c$. Hence $(f'f^*)_i$ suffers a jump at y_c unless f contains only f_1. Since $f'f^*$ is zero at both rigid boundaries and, U being monotonic, there is only one y_c, it cannot afford this jump. (If $\alpha \neq 0$, this jump corresponds to a jump in the Reynolds stress. But we do not have to consider the jump in τ, and can consider merely the jump in $(f'f^*)_i$.) Consequently, a singular neutral mode with $J(y_c)$ equal to $\frac{1}{4}$ is impossible unless f contains f_1 only [Yih, 1974b]. Hence the case of $J_c = \frac{1}{4}$ reduces to the case $J_c < \frac{1}{4}$ discussed below.

For $J(y_c) < \frac{1}{4}$ Miles [1961] gave the solutions of (126):

$$f_\pm(y) = (y - y_c)^{(1 \pm \nu)/2} w_\pm, \tag{138}$$

in which

$$w_\pm = 1 + A(y - y_c)/(1 \pm \nu) + \cdots, \tag{139}$$

with A given by (134) (but with $\frac{1}{2}$ therein replaced by $(1 \pm \nu)/2$) and

$$\nu = (1 - 4J_c)^{1/2}, \qquad J_c = J(y_c). \tag{140}$$

(Note that (138) differs from (108) only by the factor $\sqrt{\rho}$ in the definition of w. We use (138) for convenience.) We can use (138) and (139) with all terms therein considered dimensionless. Miles [1961, pp. 506 and 507] showed that for $J_c < \frac{1}{4}$ the solution of (104), if one exists, must be either f_+ or f_-. We can demonstrate our point by considering f_+ as the solution. The demonstration for the other case is strictly similar.

The study of the eigenvalue problem defined by (126) and (129) naturally leads to a study of the zeros of f. Since f is given by (138), it leads to the study of the zeros of $w_\pm$. This in turn leads us to consider the differential equation for w (from which the subscripts are removed for convenience). Denoting w_+ or w_- by w, we can easily obtain that equation:

$$(\bar{\rho} z^{2\gamma} w')' + z^{2\gamma}\left[-J_c \bar{\rho} z^{-2} + \gamma \bar{\rho}' z^{-1} + \frac{(\bar{\rho} U')'}{c - U} - \alpha^2 \bar{\rho} - \frac{N\bar{\rho}'}{(c - U)^2}\right] w = 0, \tag{141}$$

with $z = y - y_c$, and $\gamma = (1 \pm \nu)/2$.

We are now in a position to present

Theorem 1. *If $\bar{\rho}$ and U are continuous and analytic, with $\bar{\rho}' < 0$ and $U' > 0$, and if $(\bar{\rho}U')'$ and $(\ln \bar{\rho})''$ are positive throughout, then singular neutral modes are impossible.*

Proof. We have shown that it is necessary only to consider the case $J(y_c) < \frac{1}{4}$. We may consider f_+ only, since the proof for f_- is the same and since the solution is either f_+ or f_-. Now at $y = y_c$ we have $f_+ = 0$. Near y_c we have

$$Q \equiv -\frac{N\bar{\rho}'}{(U-c)^2} - \frac{J_c\bar{\rho}}{z^2} = \frac{\bar{\rho}J_c}{z^2}\left[\left(\frac{\bar{\rho}''}{\bar{\rho}'} - \frac{U''}{U'}\right)z + \cdots\right]. \tag{142}$$

Since ρ' is negative and U' and $(\bar{\rho}U')'$ are positive, U'' is positive. Thus $U - c$ is greater than $U_c'z$ for $z > 0$. On the other hand, $-\bar{\rho}'/\bar{\rho}$ is less than $(-\bar{\rho}'/\bar{\rho})_c$ for $y > y_c$, since $(\ln \bar{\rho})''$ is positive. Q is negative in the neighborhood of y_c, as can be seen from (142). Hence for $z > 0$ the term

$$-\frac{N\bar{\rho}'}{(U-c)^2}$$

is less than $\bar{\rho}J_c/z^2$ and Q is negative. Equation (142) exhibits the behavior of Q near y_c. Let the bracket in (141) be denoted by $-G$. Then since Q is negative and $U - c$ is positive for $y > y_c$, and since $\bar{\rho}'$ is negative and $(\bar{\rho}U')'$ positive, G must be positive for $y > y_c$. Multiplying (141) by w and integrating between y_c and 1, we have

$$(\rho z^{2\gamma}ww')_1 - \int_{y_c}^{1} z^2\gamma(\bar{\rho}w'^2 + Gw^2)\,dy = 0, \tag{143}$$

where the subscript 1 indicates that the parenthesis is evaluated at $y = 1$. Note that the integral in (143) is convergent in spite of the simple pole in two terms contained in G—one of which is Q, as indicated by (142). Equation (143) clearly shows that $w(1)$ cannot be zero; hence the theorem.

Another result is

Theorem 2. *If $\bar{\rho}$ and U are continuous and analytic, with $\bar{\rho}'$ negative and U' positive, and if U'' and $(\ln \bar{\rho})''$ are negative throughout, then singular neutral modes are impossible.*

The proof for this theorem is similar to that for Theorem 1. The only modification demanded for clarity is that instead of (138) we should write

$$f_\pm(y) = z^{(1\pm\nu)/2}w_\pm(z)$$

with *z now defined as $y_c - y$*. The equation corresponding to (141) is now

$$\frac{d}{dz}\left[\bar{\rho}z^{2\gamma}\frac{d}{dz}w\right] + z^{2\gamma}\left[-J_c\bar{\rho}z^{-2} + \gamma\bar{\rho}'z^{-1} + \frac{(\bar{\rho}U')'}{c-U} - \alpha^2\bar{\rho} - \frac{N\bar{\rho}'}{(c-U)^2}\right]w = 0, \tag{144}$$

in which, it must be emphasized, all primes indicate differentiation with respect to y, not z. The rest is strictly similar to the proof for Theorem 1, except in that the range of integration is between $z = 0$ and $z = y_c$ (or between $y = y_c$ and $y = 0$), and we want to show $w \neq 0$ at $y = 0$. Note also that $U'' < 0$ now guarantees $(\bar{\rho}U')' < 0$.

Since the nonexistence of singular neutral modes implies the nonexistence of unstable modes, we have also

Theorem 3. *If $\bar{\rho}$ and U are continuous and analytic, with $\bar{\rho}'$ negative and U' positive, and if either $(\bar{\rho}U')'$ and $(\ln \bar{\rho})''$ are both positive throughout, or U'' and $(\ln \bar{\rho})''$ are negative throughout, the flow is stable.*

This theorem is the natural generalization of Rayleigh's theorem for inviscid homogeneous fluids in shear flow. Previous attempts at this generalization [Synge, 1933; Yih, 1957; Drazin, 1958] have produced the result that there must be stability if (in dimensional terms)

$$\frac{2\beta g(U - c_r)}{|U - c|^2} - \frac{(\bar{\rho}U')'}{\bar{\rho}}$$

does not change sign. This criterion is not useful because it involves not only c_r but also c_i.

7.4.2. ***Sufficient Conditions for Instability.*** Sufficient conditions for instability have seldom been given in studies of hydrodynamic stability. In giving some such conditions, we shall also be able to explain why the α can be multi-valued for the same N at neutral stability.

We assume that $\bar{\rho}$ and U are analytic, that $\bar{\rho}' \leq 0$, and that at a point where $\bar{\rho}' = 0$ U'' is also zero. The value of U at that point will be denoted by U_c, for we shall consider the possibility of having c equal to U at that point. We demand that at any other point where $U = U_c$, $\bar{\rho}' = 0 = U''$ must be satisfied. If U is monotonic, of course there is only one point at which $U = U_c$.

Under the assumptions made, $\bar{\rho}''$ must be zero at y_c, since $\bar{\rho}'$ is never positive, and near y_c

$$\bar{\rho}' = \bar{\rho}_c''(y - y_c).$$

If $\bar{\rho}_c''$ were not zero $\bar{\rho}'$ would be positive for y slightly smaller or larger than y_c. With this realization, it is immediately clear that the bracket in (126) has no singularity at y_c. Let us denote the bracket in (126) by the symbol B, which is a function of y, α, and N. Then if m is the minimum of $B/\bar{\rho}$ between two points y_1 and y_2, and $0 \leq y_1 < y_2 \leq 1$, for $\alpha = 0$, and if

$$m \geq \frac{(n\pi)^2}{(y_2 - y_1)^2}, \qquad n \text{ a positive integer,} \tag{145}$$

by the use of Sturm's first comparison theorem we know that there must be n zeros of f between y_1 and y_2, whatever the value of $f(0)$ and $f'(0)$. (Note that the $\bar{\rho}$ in m or in (126) is dimensionless.) We can always choose $f(0) = 0$. If (145) is satisfied then there must be n internal zeros of f. We can increase α so that, again by Sturm's first comparison theorem,

$$f(1) = 0$$

for

$$\alpha = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n,$$

where

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n.$$

It is evident that for $\alpha = \alpha_i$ there are $n - i$ internal zeros.

Hence we have

Theorem 4. *Under the assumptions stated in the second paragraph of Section 7.4.2, if (145) is satisfied there are at least n modes with $c = U_c$ and $\alpha = \alpha_i$ ($i = 1, 2, \ldots, n$), and with α_i increasing with i. For the ith mode there are $n - i$ internal nodes.*

It is easy to show, by exactly the same approach used by Lin [1955, pp. 122 and 123], which we shall not repeat here, that by varying α^2 slightly (now not necessarily by decreasing it, as is in Lin's case), c will become complex. Hence we have

Theorem 5. *Near the neutral modes stated in Theorem 4, there are contiguous unstable modes.*

Theorem 4 explains why for the same N, given $\bar{\rho}$ and U, there can be many values for α on the neutral-stability curve (or curves), which has been observed by Miles [1963] for a special $\bar{\rho}$ and a special U.

We can sharpen Theorems 4 and 5 by defining M to be the maximum of $B/\bar{\rho}$ in (y_1, y_2) for $\alpha = 0$. Then if (145) holds and

$$M \leq \frac{(n + 1)^2\pi^2}{(y_2 - y_1)^2}, \tag{146}$$

the words "at least" in Theorem 4 can be replaced by "exactly."

We note that the analyticity of $\bar{\rho}$ and U is needed only near y_c, and that, as a consequence of Theorem 5, a layer of homogeneous fluid containing a point of zero U'' and adjoining a stratified layer with uniformly large $J(y)$ is always unstable.

7.4.3. ***Nonsingular Modes.*** Yih [1970a] also investigated waves with a (real) c outside of the range of U, whose minimum and maximum will continue to be denoted by a and b. We assume that $\bar{\rho}$ and U are continuous and that their derivatives as appear in (126) exist. Then if m and M retain their definitions as given by (145) and (146), except that $c = a - \varepsilon$, we have

Theorem 6. *Under the assumptions on $\bar{\rho}$ and U stated above, if* (145) *holds there are at least n modes with $c = a - \varepsilon$, $\alpha = \alpha_1$ $(i = 1, 2, \ldots, n)$, and α_i increasing with i. For the ith mode there are $n - i$ internal nodes. If* (146) *holds in addition, then there are exactly n such modes. If $(\bar{\rho}U')'$ is negative, then n can only increase as the arbitrary positive constant ε decreases.*

The proof of this theorem is by a straightforward application of the first comparison theorem of Sturm. Similarly, if m and M are defined by (145) and (146), except that $c = b + \varepsilon$, where ε is an arbitrary positive constant, we have

Theorem 7. *Under the assumptions on $\bar{\rho}$ and U stated above, if* (145) *holds there are least n modes with $c = b + \varepsilon, \alpha = \alpha_i (i = 1, 2, \ldots, n)$, and α_i increasing with i. For the ith mode there are $n - i$ internal nodes. If* (146) *holds in addition, then there are exactly n such modes. If $(\bar{\rho}U')'$ is positive, then n can only increase as ε decreases.*

If for $c = a - \varepsilon$ or $c = b + \varepsilon$, and any $\varepsilon \geq 0$, M is less than $\pi^2/(y_2 - y_1)^2$ for all y_1 and y_2 between zero and 1, then there can be no waves propagating with c equal to a or b, or outside of the range of U. On the other hand, if $U' = 0$ at the point of maximum or minimum U, and, *a fortiori*, if there is a region of constant U where $U = a$ or b, it can be easily shown that waves of any finite wave length and any finite number of internal modes n can propagate with $c \leq a$ or $c \geq b$. All this is in contrast with waves propagating in a layer of homogeneous fluid with a free surface and in shear flow. In that case [Yih, 1970a], if U is monotonically increasing with y, waves of all wave lengths can propagate with c greater than b, and only sufficiently long waves can propagate with c less than a.

7.5. *Instability as a Result of Resonance*

In giving special results on instability of stratified flows, Taylor [1931b] already remarked vaguely on the cause of instability as two wave trains resonating with each other. The precise meaning of his idea was already evident upon examination of the results for Helmholtz instability, i.e., (71) and (72). When the quantity within the braces of either of these equations is negative, the stratified flow of Helmholtz is unstable. When that quantity is positive, there are two neutral wave trains with different wave velocities. When that quantity is zero, these two wave velocities coalesce, the flow is marginally stable, and instability is imminent. Thus the instability can be considered to be caused by the resonance of two wave trains.

This interpretation is probably generally valid and can be generalized to include all modes (when there is more than one) of the perturbation. We shall now give this generalization [Yih, 1974b] by considering (and therefore under the restriction of) a class of two-dimensional flows characterized by (i) a middle layer with constant density and linear velocity, (ii) a stably stratified upper layer with a constant velocity, (iii) a stably stratified lower layer with a constant velocity, and (iv) the density and the velocity are continuous throughout. We shall discuss the stability of this flow in a general way, and reach the conclusions that instability is the result of a kind of resonance, that there are infinitely many modes which, in the neutral cases at least, are characterized by the number of internal zeros, and that for the flows considered here gravity is always *destabilizing*.

It seems that the conclusions to be drawn for the class of flows being considered will lead to a better understanding of the stability of stratified flows, for which the neutral curves, such as those given by Miles [1963], seem to defy any intuitive interpretation. Numerical results will be given for exponential stratifications in the upper and lower layers, with the simplification afforded by the Boussinesq approximation.

Stability of stratified flows in general not subject to condition (i) is then considered, and it is shown that neutral normal modes which are continuations of complex-conjugate unstable and damped modes across the stability boundary do not exist.

The indices 1, 2, and 3 are assigned to the lower, middle, and upper layers, respectively, and the depths of the layers will be denoted by d_1, d_2, and d_3. For convenience let

$$d_2 = 2d.$$

Then, the velocity in the middle layer is Vy/d, the velocity in the upper layer is V, and that in the lower layer is $-V$. The density in the middle layer is constant and is denoted by ρ_0. The mean density $\bar{\rho}$ in the upper or the lower layer is such that $\bar{\rho}' < 0$, but is otherwise unspecified until the numerical example is discussed.

In the variables defined by (125), after the carets are dropped, the dimensionless stream function for the perturbation is

$$\psi = f(y) \exp i\alpha(x - ct),$$

and we obtain, from (93) and (94), with ρ' now denoting ρ/ρ_0 (recall that ρ is the density perturbation)

$$(u, v, \rho') = \left(f', -i\alpha f', \frac{1}{U - c}\bar{\rho}'\right) \exp i\alpha(x - ct). \tag{147}$$

The perturbation pressure is given by (90):

$$p = \bar{\rho}(cf' - Uf' + U'f) \exp i\alpha(x - ct). \tag{148}$$

The differential equation is (126).

For the mean velocity distribution specified, the interfacial conditions are, since U and $\bar{\rho}$ are continuous, (a) the continuity of f and (b) the continuity of p. In dimensionless terms, the mean velocity is given by

$$\begin{aligned} &U = 1 \quad \text{for} \quad y \geq 1, \qquad U = y \quad \text{for} \quad |y| \leq 1, \\ &U = -1 \quad \text{for} \quad y \leq -1. \end{aligned} \tag{149}$$

The interfacial conditions are

$$\begin{aligned} f_1(-1) &= f_2(-1), \\ (c+1)f'_1(-1) &= (c+1)f'_2(-1) + f_2(-1), \end{aligned} \tag{150}$$

$$\begin{aligned} f_2(1) &= f_3(1), \\ (c-1)f'_3(1) &= (c-1)f'_2(1) + f_2(1). \end{aligned} \tag{151}$$

The boundary conditions at the rigid boundaries are

$$f_1(-1-a) = 0, \qquad f_3(1+b) = 0, \tag{152}$$

where

$$a = d_1/d, \qquad b = d_3/d. \tag{153}$$

We shall use the constant density ρ_0 in the middle layer to be the density scale, so that the dimensionless density in the middle layer is 1, or

$$\bar{\rho} = 1 \qquad \text{for} \quad |y| \le 1. \tag{154}$$

First, we note that at the interfaces $\bar{\rho}$ and U are continuous but U' is discontinuous. Integration of (126) in the Stieltjes sense across the interfaces produces the second conditions in (150) and (151), so that these are *natural* conditions, and the Sturm-Liouville theory can be applied to (126) and (152), without concern about (150) and (151), so long as $|c| \ne 1$, so that (126) is not singular. We shall now apply that theory.

We shall consider real values of c with $|c| < 1$, for Howard's semicircle theorem tells us that the neutral modes contiguous with the unstable ones must have their c satisfying $|c| \le 1$, and (110) rules out the extreme values of U as possible values for c.

First we note that for $N = 0$ (zero gravity) there may be one or two positive values of α^2 which enable a solution of (126) to satisfy the boundary conditions and interfacial conditions, since the term in (126) containing $(\rho U')'(c - U)^{-1}$, considered as a generalized function at the interfaces, may have, for the U specified by (149), the effect of making the coefficient of f in (126) positive when (126) is considered as the equation for the entire flow. There can be no more than two modes for $N = 0$, the first having no internal node and the second only one. For if there were a third mode it would, according to the Sturm-Liouville theory, have two internal nodes, and either one of them would be in the bottom and top layers, or both would be in the middle layer. In either case the eigenfunction f in (126) would have two nodes in *one* of the three layers (remembering that f is zero on the solid boundaries), and this is quite impossible, since $N = 0$, and the coefficient of f in (126) is always negative between these nodes because no interface is crossed. We are mainly interested in the neutral curves in the N-α plane. All that we have said above means that, for the $\bar{\rho}$ and U specified in this section, there may be one or two values of α (positive by choice) at which the neutral curves intersect the axis $N = 0$.

For higher modes we can always increase N. With the understanding that n may be greater than 2 for small α^2, for any given α and for a specified number $(n - 1)$ of internal zeros of the eigenfunction, we can always choose an N and *two* corresponding c's so that the boundary conditions (152) are satisfied by f, the solution of (126), for we can make

$$-\frac{N\bar{\rho}'}{(U - c)^2}$$

for the bottom or the top layers as small as we please by choosing N small, and as large as we please by choosing $|c|$ very near 1. Let the N so chosen be denoted by N_n, and the two corresponding c's be denoted by c_{n1} and c_{n2}, the n always indicating the mode. Then, c_{n1} and c_{n2} are functions of N_n.

Still fixing n and α at any given value, we now increase N_n continuously. The Sturm-Liouville theory applied to (126) then leads to the conclusion that at a sufficiently large N_n no real c_n with $|c_n| < 1$ can exist. At some N_n the two eigenvalues c_{n1} and c_{n2} must coalesce and for greater values of N_n the eigenvalues of c must become complex. When c_{n1} coalesces with c_{n2} the pair of values (α, N_n) must be on the stability boundary in the α-N plane, or the neutral curve. The coalescence of c_{n1} with c_{n2} (to make c_n a double eigenvalue) is a sort of resonance, for it occurs when two wave trains have the same wave velocity in a shear flow.

Note that for any given α, n, and an N_n however small, we can always choose c_n such that

$$1 \gg 1 - |c_n|^2 > 0,$$

in order to satisfy (126) and (154). There are two values, c_{n1} and c_{n2}, for c_n. As n increases, c_{n1} is nearer and nearer c_{n1} and c_{n2} nearer and nearer -1. As N_n is allowed to increase, c_{n1} will coalesce with c_{n2} for a value of N_n, as already explained. Thus, for any α and any integral value of n from 1 onward, there is a N_n for which the flow is neutrally stable. That is to say, for any α there are infinitely many eigenvalues of N for neutral stability, and, correspondingly, infinitely many neutral modes.

The fact that for any n the value of N_n must increase to attain complex values of c_n shows that gravity is always destabilizing for the specified U and $\bar{\rho}$.

For more general distributions of U and $\bar{\rho}$ the situation is complicated by the following facts:

(a) If $\bar{\rho}'$ and U'' are not zero at the place where $U = c$, (126) is singular.

(b) Then if the Richardson number at $U = c$ is less than $\frac{1}{4}$, the eigenfunction, if one exists, must be one or the other of the two independent solutions of (126).

(c) As N is allowed to increase, there will be a limit for N beyond which the Richardson number is everywhere greater than $\frac{1}{4}$, if $\bar{\rho}' < 0$ everywhere.

Item c indicates that if $\bar{\rho}' < 0$ everywhere there cannot be infinitely many neutral modes. There may be a few, or only one such mode, or none. The role of gravity will no longer be always stabilizing, as also indicated by the solutions of Miles [1963].

7.5.1. *An Example.* For an example, we shall let

$$a = 2 = b,$$

and

$$\bar{\rho} = \begin{cases} \rho_0 \exp\left[-\beta(y-1)\right] & \text{for the top layer,} \\ \rho_0 & \text{for the middle layer,} \\ \rho_0 \exp\left[-\beta(y+1)\right] & \text{for the bottom layer,} \end{cases}$$

where y is in units of d, the half depth of the middle layer. Then, with U still specified by (149), and with f_1, f_2, and f_3 for f in the bottom, middle, and top layers, respectively, (126) has the forms

$$f_1'' - \beta f_1' - \left(\alpha^2 - \frac{\beta N}{(1+c)^2}\right) f_1 = 0, \tag{155}$$

$$f_2'' - \alpha^2 f_2 = 0, \tag{156}$$

$$f_3'' - \beta f_3' - \left(\alpha^2 - \frac{\beta N}{(1-c)^2}\right) f_3 = 0. \tag{157}$$

The interfacial conditions are still (150) and (151), and (152) now becomes

$$f_1(-3) = 0, \qquad f_3(3) = 0. \tag{158}$$

The differential system consisting of (150), (151), and (155) to (158) define an eigenvalue problem. For given α, β, and N one can determine c from this system.

For simplicity, we shall adopt the Boussinesq approximation and write (155) and (157) as

$$f_1'' - \left(\alpha^2 - \frac{\beta N}{(1+c)^2}\right) f_1 = 0, \tag{155a}$$

$$f_3'' - \left(\alpha^2 - \frac{\beta N}{(1-c)^2}\right) f_3 = 0. \tag{157a}$$

With the Boussinesq approximation, we can show that if c is an eigenvalue, so is $-c$. This is shown by making the transformation

$$\hat{y} = -y, \qquad \hat{f}_1(\hat{y}) = f_3(y), \qquad \hat{f}_3(\hat{y}) = f_1(y), \qquad \hat{f}_2(\hat{y}) = f_2(y). \tag{159}$$

Substituting (159) into the system consisting of (150), (151), (155a), (156), (157a), and (158), and then dropping the carets, we regain that system, except that c is replaced by $-c$. Thus, if c is an eigenvalue, so is $-c$.

This fact makes it clear that for any neutral mode (not necessarily corresponding to a stability boundary), and in the notation of the last section,

$$c_{n1} = -c_{n2}. \tag{160}$$

Thus, for any mode, to obtain the relationship between α and βN (which are the only two parameters left aside from c) at the stability boundary, which corresponds to a double root in c, we can simply put c equal to zero, which is the value that an eigenvalue for c of multiplicity 2 must take. This simplifies matters a great deal. Putting c equal to zero and solving (155a), (156), and (157a) with interfacial conditions (150) and (151) and boundary conditions (158), we obtain the results,

$$(\alpha^2 - J)^{1/2} \coth 2(\alpha^2 - J)^{1/2} = 1 - a \tanh \alpha \tag{161}$$

and

$$(\alpha^2 - J)^{1/2} \coth 2(\alpha^2 - J)^{1/2} = 1 - \alpha \coth \alpha, \tag{162}$$

representing two families of the stability boundary, with

$$J = \beta N. \tag{163}$$

Equation (161) corresponds to $f_2(y)$ being an even function of y and (162) corresponds to $f_2(y)$ being odd. Figure 49 shows the variation of J with α on the stability boundary. The curve for the nth mode for (162) is indicated by the number n. Only three odd modes are shown. The curves for the nth mode for (161) are indicated by n'. Again only three modes are shown.

All the conclusions of the last section are borne out. There are infinitely many modes, gravity is always destabilizing for the assumed U and $\bar{\rho}$, and resonance is the cause of instability. At a point in the α-J plane above the curves 1 and 1′ but below 2′, there are two unstable modes, two damped modes, and infinitely many neutral modes. The enumeration of modes at other points is similar.

Note that although for $N = 0$, (155a) and (157a) have the same form as (156), and all these have exp $(\pm ay)$ for solutions, the boundary conditions (158) can be satisfied because there are two interfaces at which U' is discontinuous, and (150) and (151) must be applied.

For each mode J increases with α. For any given α there are infinitely many values of J, but for any J there are only a finite number of values for α. As expected, increasing α stabilizes the flow.

In addition to the modes separated by the stability boundaries (161) and (162), there are two other infinite sets of real c-values outside of the range of U. These are the neutral modes not contiguous to unstable modes, discussed in Section 7.4.1.

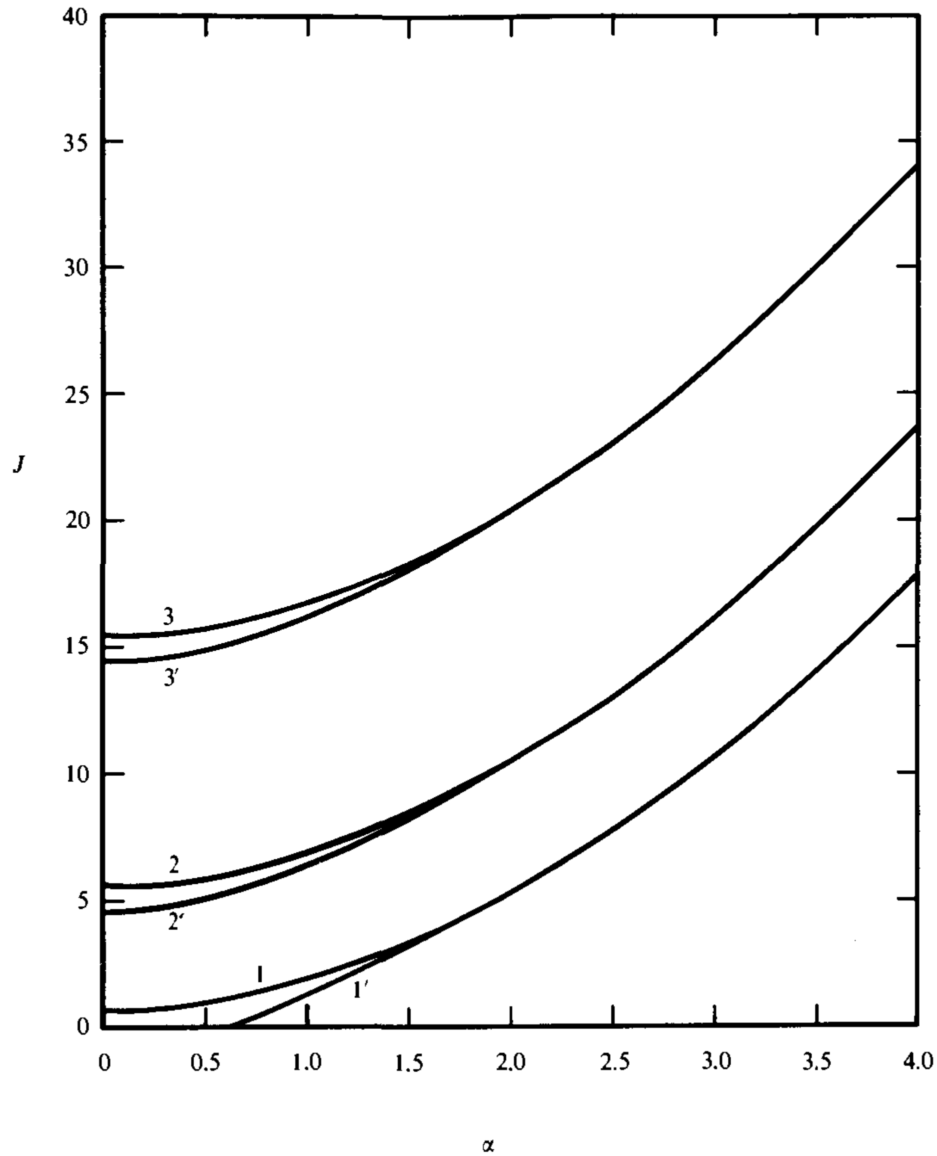


FIGURE 49. Stability boundaries in the α-J plane for various modes. The nth mode for an odd eigenfunction is indicated by the number n, and the nth mode for an even eigenfunction is indicated by n'. (From Yih [1974b].)

So far we have discussed only the density and velocity distributions specified by (149), and given an example for a special density distribution. The arguments just after (154), by means of the Sturm-Liouville theory, clearly remain valid even if there are no regions of constant velocities, provided there is a region of linear velocity and constant density, and provided there are neutral modes, with their c or c_r in this region, which coalesce in the manner described. Of course, if there are no regions of constant velocity, there cannot be infinitely many unstable modes because there cannot be infinitely many cases of coalescence, because the term containing N in (126) cannot be made arbitrarily large in the entire upper or lower layer by choosing any c.

7.6. A Discussion of Singular Neutral Modes

Miles [1961] showed that a singular neutral mode cannot exist if the value of the Richardson number J at $y = y_c$, where $U = c$ denoted by J_c, is greater than $\frac{1}{4}$. For $J_c < \frac{1}{4}$, he showed that the solution must be either of the f_+ or f_- given by (138). For $J_c = \frac{1}{4}$, as shown in Section 7.4, f must be the f_1 given by (132), apart from a multiplicative constant. Thus we can concentrate on the case $J_c < \frac{1}{4}$, for the treatment of the case $J_c = \frac{1}{4}$ is strictly similar.

If $U = c$ somewhere in the field of flow, that is, for some y in the range $0 \leq y \leq 1$ occupied by the flow, (126) is singular unless $(\bar{\rho}U')'$, $\bar{\rho}'$, and $\bar{\rho}''$ are all zero. Considering only the singular cases, for $J_c < \frac{1}{4}$ we can let $f = f_+$, for the case $f = f_-$ can be treated in a strictly similar way. Then since y_c cannot be the end fronts $y = 0$ or $y = 1$, from (138) it follows that for a fixed α (and specified $\bar{\rho}$ and U, of course)

$$w_+(0, c, N) = 0 \qquad \text{and} \qquad w_+(1, c, N) = 0, \tag{164}$$

in which c is real. Now this can represent a point on the stability boundary only if the solution of (164) gives a real c and a real and positive N. If so, then the pair of values of α (the one chosen) and N (solution of (164)) gives a point (α, N) on the stability boundary, on one side of which complex values of c exist, so that that side is the unstable region. There may be n sets of acceptable (real c, positive N) solutions of (164), in which case there are n modes of the disturbance, and an unstable region for one mode can be a stable region for another. All this is in agreement with existing results.

Consider, however, a flow in which $\bar{\rho}'$ is everywhere negative and U' finite. For this flow, at sufficiently large N the Richardson number is everywhere larger than $\frac{1}{4}$. Hence in the α-N plane there must be a region of stability. If, then, there is an unstable region for any mode, there is for that mode a stability boundary, to one side of which the flow is unstable for that mode of disturbance and to the other side of which it is stable.

The question then arises: What happened to the unstable mode as one crosses the stability boundary on which that mode is a singular neutral mode? It cannot be continued into other singular neutral modes with slightly changed (but real) c when N is slightly varied for a fixed α, since *all* the singular neutral modes are given by (164) and are accounted for as the stability boundaries for the various modes. It cannot be continued into a nonsingular neutral mode with a c outside of the range of U, since the nonsingular neutral modes form a set of their own and have nothing to do with the singular neutral modes.

The resolution of this dilemma necessitates separating the damped modes from the amplified ones, thus abandoning the notion that these modes coexist, with complex conjugate values for c, in the unstable region. Then when the stability boundary is crossed the unstable mode continues into a stable mode. The argument used by Lin [1955, pp. 122, 123] for the homogeneous fluid and by Howard [1963, pp. 338–342] for a nonhomogeneous fluid to show how unstable modes are created by varying the wave number at neutral stability, albeit not rigorous mathematically, does provide the resolution required. The justification for this argument must eventually be sought in the effects of viscosity.

8. THE DISH-PAN EXPERIMENTS

Experiments on gravitational convection in a rotating fluid have been carried out by Fultz [1953] and Hide [1953b]. Fultz's experiments were originally carried out in a dish pan, to study large-scale circulation in the atmosphere, and Hide's experiments were conducted with a view to explaining the origin of terrestrial magnetism. The papers referred to contain good bibliographies for the literature prior to 1953. A good deal of work has been done by these authors and by others since then. For the literature since 1953, see Fultz [1958 and 1962].

In both Hide's and Fultz's experiments a rotating circular pan is cooled at the center and heated at the rim. If the radius of the pan and that of an inner cooling drum are fixed, the convection pattern depends on the value of the parameter $-d\Delta\rho/\rho\omega^2$, in which d is the depth of the fluid, ρ a reference density, $\Delta\rho$ a reference density difference (say between the densities at the outer and inner boundaries), and ω the angular speed of the pan, the physical properties of the fluid being approximately fixed. The smaller the value of the parameter, the more complicated the convection pattern, which can be axisymmetric, then have two petals, three petals, etc. Kuo [1953] assumed the temperature to be a linear function of the radial and vertical coordinates, neglected certain terms in the governing differential equation, and for approximate (for the free-surface case only) boundary conditions solved the differential system to find the criterion for the onset of axisymmetric convection. In spite of the fact that the basic temperature (and, by implication, the basic flow) has been assumed and that some simplifying assumptions have been made, Kuo's theoretical results agree with Fultz's experimental ones quite well.

NOTES

Section 5

1. If one adopts the Boussinesq approximation to study the Kelvin-Helmholtz instability of a vortex sheet, with an antisymmetric (and discontinuous) velocity distribution, one sees readily that for unstable or neutral modes the real part c_r of the complex velocity c is zero. Howard [1963]

considered the case of two vortex sheets, retaining the Boussinesq approximation, and found that c_r is not zero on the stability boundary, that nevertheless there are only nongrowing waves with $c_r = 0$ inside the stable region, and that there is a region with $c_r = 0$ inside the unstable region. All this points to the danger of setting $c_r = 0$ for antisymmetric velocity profiles and density profiles in order to find the stability boundary. This danger does not exist for the single shear layer treated in Sec. 7.5.

2. Finite-amplitude Kelvin-Helmholtz billows have been investigated by Maslowe and Kelly [1970] and Maslowe [1973].

Section 7

1. Howard [1964] studied the number of unsteady modes of the Rayleigh equation by means of the Sturm-Liouville theory, for flows with mean velocity $U(y)$ satisfying the equation

$$U''(y) = K(y)(U - c_s),$$

where $K(y)$ is some continuous or piecewise continuous function of y, and $U - c_s$ is assumed to vanish at least once in the domain of flow. If the domain extends to infinity K is assumed to approach zero there. Many interesting and useful results are obtained. This work does not deal with stratification, and yet it is very relevant to the approach used in Section 7.

2. Kelly and Maslowe [1970] have shown that nonlinearity tends to reduce the local Richardson number at the critical layer of a slightly stratified shear flow.

3. Wang [1975] investigated experimentally the stability of shear flows of a stratified fluid. The stability boundary for the velocity and density distributions under study had been provided by Hazel [1972]. The stability boundaries in the plane of the overall Richardson number J and the wave number α (of the disturbance), obtained numerically by Hazel are

(a) a straight line L starting from the origin and making an angle of less than 45° with the J-axis, and
(b) a curve C ascending (in J) from the point $(\alpha, J) = (0, 1)$ as α increases.

According to Hazel's theory, between the J-axis and L (Region I) the flow is stable. Between L and C (Region II) it is unstable. Between C and the α-axis it is stable. Wang's experiments also gave a straight stability boundary, but it is well within Region I, not coinciding with L. They also gave another stability boundary, corresponding to C, which is, however, well within Region II, far from C—indeed rather near L. Hence the agreement between Hazel's theory and Wang's experiments is not satisfactory, although the general trend of Hazel's stability boundaries are verified.

The disagreement might be explained in the following way. Wang's unstable region (between two stability boundaries) invades Hazel's upper stability region because of finite-amplitude effects of the disturbance. Wang's lower stability boundary lies in Hazel's unstable region because of viscous effects. But these are only possible, unsubstantiated explanations and could also be used to explain away the disagreement if Wang's curves had shifted the other way (i.e., clockwise from Hazel's).

4. The instability of the flow of a fluid stratified in viscosity was studied by Yih [1967a], who found that viscosity stratification can cause instability. The role of viscosity stratification in the stability of layered flows has been investigated further by Kao [1968], Li [1969], and Craik 1969].

5. Blumen [1975] studied, mainly by numerical computation, the stability of some stratified flows for which the mean velocity in the x-direction depends not only on the vertical coordinate z but also on the lateral coordinate y. No surprising results were obtained. Nevertheless the results are good to have on record.

6. The stability of a vertically stratified viscous fluid set in motion by an oscillating vertical plane was studied by von Kerczek and Davis [1976], who showed that the flow is highly unstable when the forcing frequency and the internal-wave frequency of the fluid are not far apart. The stability of the same flow was studied by Bergholz [1978] for a different range of the parameters involved.

Section 7.4.3

1. The shape and breaking of nonsingular neutral waves of finite amplitude in stratified shear flows have been studied by Thorpe [1978].

General

1. Somewhat analogous to the phenomenon of energy transfer among wave triads, first studied by Phillips [see Phillips, 1966, for references], are the instability of Stokesian waves [Benjamin and Feir, 1967] and the instability of oscillating internal waves, studied by Davis and Acrivos [1967b], The instability of internal waves found by Davis and Acrivos depends on the availability of higher modes to satisfy the requisite conditions, which are that the difference of the wave numbers of two wave trains equals the wave number of the third and that the same is true of the frequency. (The wave train whose stability is being studied is one of these wave trains.) Without these higher modes (with more nodal planes) these conditions cannot be met for free waves.

However, if one wave train is generated by a flow over a wavy bottom, it is always unstable if the appropriate Froude number of the flow is greater than 1, or if the wave number of the basic wave train is sufficiently high for any Froude number, as was found by Yih [1976b], who investigated free-surface waves, internal waves in two superposed layers, or internal waves in a continuously stratified fluid.

Chapter 5

FLOWS IN POROUS MEDIA

I. INTRODUCTION

Fluid flow in porous media is an important subject in hydrology and is of vital interest to the petroleum industry. The law governing seepage flow of a homogeneous fluid in a homogeneous and isotropic porous medium was given by Henry P. G. Darcy in his Paris treatise [1856]. This law, originally given for one-dimensional flow only, can be generalized to apply to three-dimensional flows of a heterogeneous fluid. In the notation of Chapter 1, the generalized Darcy equations are

$$\frac{\mu}{k} u_i = -\frac{\partial p}{\partial x_i} + \rho X_i \qquad (i = 1, 2, 3), \tag{1}$$

in which μ and ρ are the viscosity and density of the fluid, k is the permeability of the medium, p is the pressure, and X_i is the ith component of the body force per unit mass. If the direction of increasing Z is opposite that of the gravitational acceleration g, and if the fluid is homogeneous, (1) can be written as

$$\frac{\mu}{k}(u, v, w) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(p + \rho g Z), \tag{1a}$$

in which u, v, and w are used for u_1, u_2, and u_3, and x, y, and z replace x_1, x_2, and x_3. Thus the flow is irrotational if μ and k are constant as well as ρ. The pressure p in this equation may be considered to be the mean of the three normal stresses. But since Darcy's law is valid only for very slow flows, the three normal stresses do not differ materially from one another. Hence they do not differ materially from p. Thus, on a free surface, instead of specifying the normal stress to be zero, one can simply set p equal to zero.

The equation of continuity for an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2}$$

For the flow of a homogeneous fluid in a homogeneous medium, μ and k are constants, and (1) can be combined with (2) to form the single equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0, \tag{3}$$

in which

$$\phi = -\frac{k}{\mu}(p + \rho g Z)$$

is a velocity potential, the gradient of which is the velocity. Thus, if the boundary conditions can be specified in terms of ϕ alone, (3) can be solved, at least in principle, and the effect of gravity is merely to change the pressure by an amount in accordance with the law of hydrostatics. In such cases the boundaries are fixed, so that for flows in domains with rigid boundaries the effect of gravity is rather trivial. However, if there exist a free surface, one boundary condition on that free surface is

$$\phi = -\frac{k}{\mu}\rho g Z. \tag{4}$$

Gravity now plays an important role in the determination of the solution. In this chapter, the discussion of the flow of a homogeneous fluid is limited exclusively to flows with a free surface or an interface. Near the end of this chapter seepage flows of a nonhomogeneous fluid will be discussed, in which gravity plays an important role even if the boundaries are fixed.

1.1. *Darcy's Law Further Generalized*

In (1) the term on the left-hand side represents the viscous resistance. If we assume this to be true even if inertia effects are not neglected, Darcy's law becomes

$$\frac{\rho}{\varepsilon}\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho X_i - \frac{\mu}{k}u_i, \tag{5}$$

in which ε is the porosity, and D/Dt stands for substantial differentiation. Equation (5) may be regarded as Darcy's law in its most general form. Of course the convective acceleration is too small to matter, but the temporal acceleration should be included in unsteady-flow problems, and, strictly speaking, even in stability problems. In the general discussion that follows, the convective acceleration will be included, for the sake of clarity.

If the body force is assumed to have a potential gZ, and ρ is constant, (5) can be written as

$$\frac{\rho}{\varepsilon}\frac{Du_i}{Dt} + \frac{\mu}{k}u_i = -\frac{\partial}{\partial x_i}(p + \rho g Z). \tag{5a}$$

If, further, μ, k, and ε are constant, cross differentiation of (5a) yields

$$\frac{\rho}{\varepsilon}\frac{D\xi_i}{Dt} = \xi_\alpha \frac{\partial u_i}{\partial x_\alpha} - \frac{\mu}{k}\xi_i,$$

if the equation of continuity $\partial u_\alpha/\partial x_\alpha = 0$ is used. For two-dimensional flows

$$\frac{\rho}{\varepsilon}\frac{D\zeta}{Dt} = -\frac{\mu}{k}\zeta, \qquad \text{with} \quad \zeta = \xi_3,$$

which means that the vorticity ζ of a particle will decrease exponentially as

$$\exp\left(-\frac{\mu}{\rho}\frac{\varepsilon}{k}t\right),$$

if the physical constants do not vary in the field of flow.

Results of greater generality can be obtained by considering the circulation

$$\Gamma = \oint u_i\, dx_i.$$

Consider a material circuit, along which dx_1, dx_2, and dx_3 are measured at any time. Since

$$\frac{D\, dx_i}{Dt} = du_i,$$

it follows from (5a) that

$$\frac{\rho}{\varepsilon}\frac{D}{Dt}\oint u_i\, dx_i + \frac{\mu}{k}\oint u_i\, dx_i = -\oint d\left(p + \beta g Z - \frac{\rho}{\varepsilon}\frac{u_i u_i}{2}\right) = 0,$$

since p and u_i are single-valued. Thus

$$\frac{\rho}{\varepsilon}\frac{D\Gamma}{Dt} + \frac{\mu}{k}\Gamma = 0,$$

and

$$\Gamma = \Gamma_0 \exp\left(-\frac{\mu\varepsilon t}{\rho k}\right).$$

Hence the circulation along *any* material circuit will be zero if it was originally zero, and will otherwise tend to zero as a limit. Irrotationality will therefore persist in a flow if it exists for a moment throughout the fluid, and will otherwise emerge as the final state of flow. Thus steady flows are irrotational if ε, ρ, μ, and k are constant.

Since convective acceleration, being quadratic in u_i, is very small, for steady flows there is no need to use (5) instead of (1). For unsteady flows it is not obvious that the temporal acceleration can always be neglected. In Section 6 the effect of temporal acceleration on the solution of a stability problem will be discussed.

2. STEADY TWO-DIMENSIONAL FLOWS WITH A FREE SURFACE

The flow of a homogeneous fluid in an isotropic medium with constant permeability is governed by (3). If the flow is steady and two-dimensional, the free surface must be a streamline on which (4) is satisfied. If, for convenience, the directions of increasing y and Z are chosen to coincide, (4) assumes the form

$$\phi = -\xi_0^{-1} y, \tag{4a}$$

with

$$\xi_0 = \frac{\mu}{k\rho g}.$$

The form of the free surface is, of course, not known *a priori*. Hence two boundary conditions must be specified for the free surface. One is (4a); the other is

$$\psi = \text{constant} \tag{6}$$

on the free surface, in which ψ is the harmonic conjugate of ϕ, and stands for the stream function.

Now (4a) is a very inconvenient boundary condition to apply, because neither ϕ nor y is known on the free surface. This is the reason why there exist so very few exact solutions for free-surface flows in porous media. But for two-dimensional flows there are at least two useful methods which can be employed to attack such problems, that is, the hodograph method and the inverse method. These methods will be presented in this section through some examples which are interesting in themselves. Possible variations of the methods are not presented, and the examples given are far from being exhaustive. For a more extensive treatment of free-surface seepage flow, the readers are referred to standard books on porous media.

2.1. *Hodograph Method*

If s is the distance and q the speed along the free streamline, application of Darcy's law in the direction of increasing s yields

$$q = \frac{\partial \phi}{\partial s}. \tag{7}$$

If (4a) is differentiated with respect to s and the result multiplied by q, the equation

$$q^2 = -\frac{q}{\xi_0}\frac{\partial y}{\partial s} \tag{8}$$

is obtained. Now since the velocity on the free surface is tangent to it (because the flow is steady), and has the magnitude q, it follows that

$$q\frac{\partial y}{\partial s} = v \qquad \text{and} \qquad q^2 = u^2 + v^2,$$

and (8) becomes

$$u^2 + v^2 = -\frac{v}{\xi_0}. \tag{9}$$

Thus the free surface is a circular arc in the hodograph plane. This fact has been known for some time, and in Muskat's book [1937] it has been utilized to solve the problem of seepage flow, with a free surface, through a rectangular earth dam. The solution involves elliptic modular functions. For details, see Muskat's book, and for numerical evaluation of Muskat's solution, see Mahdaviani [1962].

The velocity components are related to ϕ by

$$u = \frac{\partial\phi}{\partial x}, \qquad v = \frac{\partial\phi}{\partial y},$$

and the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

again allows the use of a stream function ψ, in terms of which u and v can be expressed

$$u = \frac{\partial\psi}{\partial y}. \qquad v = -\frac{\partial\psi}{\partial x}.$$

Thus, as in Section 14 of Chapter 3, the complex potential

$$w = \phi + i\psi$$

can be used. Since

$$\frac{dw}{dz} = u - iv, \tag{10}$$

u and $-v$ are harmonic conjugates, as w is an analytic function of z as well as $u - iv$. Since the hodograph of the free surface is given by (9), the region in the hodograph plane is definite if the other boundaries (if any) are straight, and in principle the problem can be solved. But the circular hodograph, though definite, is still rather inconvenient to deal with. One of the two key steps used in the solution of free-surface flows in porous media is the one from (4a) to (9). The other is the following simple transformation:

$$\zeta = \frac{i}{dw/dz} = \frac{i}{u - iv} = \xi + i\eta. \tag{11}$$

Since

$$\zeta = \frac{-v + iu}{u^2 + v^2}, \tag{12}$$

(9) shows that on the free surface

$$\xi = \xi_0. \tag{13}$$

Thus the free surface has been transformed into a straight line.

Equation (13) has been derived from (4a) and (6). Another condition on the free surface is needed so that (4a) and (6) can be adequately replaced. Since on the free surface $dw = d\phi$, differentiation of (4a) yields

$$dw = -\frac{1}{\xi_0}\,dy,$$

which, in view of (11), can be written as

$$\frac{i}{\zeta}\,dz = -\frac{1}{\xi_0}\,dy,$$

or

$$\frac{1}{\zeta}\frac{dz}{d\zeta} = \frac{i}{\xi_0}\frac{dy}{d\zeta}$$

on the free surface. Since on the free surface ξ = constant, there $d\zeta = id\eta$. Hence

$$\operatorname{Im}\frac{1}{\zeta}\frac{dz}{d\zeta} = 0. \tag{14}$$

This is the condition sought. The essence of the hodograph method lies in the utilization of the complex variable ζ and the conditions (13) and (14) on the free surface. Two examples of some practical interest will be presented in the next two subsections.

2.2. *Free-Surface Flow into a Sink*

If a sink is situated at the origin, then *near the origin*

$$w \sim -m \ln z,$$

in which $2\pi m$ is the total discharge (per unit length) into the sink. Hence near the origin

$$\zeta \sim -\frac{iz}{m}$$

and

$$\frac{1}{\zeta}\frac{dz}{d\zeta} \sim mi\zeta^{-1} \qquad \text{near} \quad z = 0. \tag{15}$$

But (14) demands that $(1/\zeta)(dz/d\zeta)$ be real on $\xi = \xi_0$. Hence

$$\frac{1}{\zeta}\frac{dz}{d\zeta} = mi\left(\frac{1}{\zeta} - \frac{1}{2\xi_0 - \zeta}\right), \tag{16}$$

or

$$\frac{dz}{d\zeta} = mi\left(1 - \frac{\zeta}{2\xi_0 - \zeta}\right) = 2mi\left(1 + \frac{\xi_0}{\zeta - 2\xi_0}\right).$$

One quadrature of this equation yields

$$z = 2mi\left(\zeta + \xi_0 \ln \frac{2\xi_0 - \zeta}{2\xi_0}\right), \tag{17}$$

the constant of integration being so determined as to make z approach zero as ζ approaches zero. Equation (17) gives the solution to the problem. Upon division of it by $m\xi_0$, it assumes the dimensionless form [Yih, 1964]

$$z_1 = 2i\left(\zeta_1 + \ln \frac{2 - \zeta_1}{2}\right), \tag{18}$$

in which

$$z_1 = \frac{z}{m\xi_0} = x_1 + iy_1, \qquad \zeta_1 = \frac{\zeta}{\xi_0} = \xi_1 + i\eta_1.$$

On the free surface $\xi = \xi_0$, hence $\xi_1 = 1$. Thus (18) can be resolved into

$$\frac{x_1}{2} = -\eta_1 + \tan^{-1} \eta_1 \tag{19}$$

and

$$\frac{y_1}{2} = 1 + \ln \sqrt{\frac{1 + \eta_1^2}{2}}, \tag{20}$$

which are parametric equations of the free surface. The cusp point at which the free surface streamlines $\psi = 0$ and $\psi = m$ meet is given by

$$\frac{dx_1}{dy_1} = 0, \qquad \text{or} \quad \frac{dx_1/d\eta_1}{dy_1/d\eta_1} = 0.$$

Application of L'Hôpitals' rule to this equation yields $\eta_1 = 0$, which gives $x_1 = 0$ and $y_1 = 2(1 - \ln 2)$ for the coordinates of the cusp point.

The other streamlines are determined by writing (18) as

$$\frac{x_1}{2} = -\eta_1 + \tan^{-1} \frac{\eta_1}{2 - \xi_1} \tag{21}$$

and

$$\frac{y_1}{2} = 1 + \ln \sqrt{\frac{(2 - \xi_1)^2 + \eta_1^2}{2}}. \tag{22}$$

For a fixed x_1 assume various values of η_1. Compute ξ_1 from (21), then y_1 from (22). Thus, $\xi_1 + i\eta_1$ for many values of y_1 for a fixed x_1 are obtained.

Since

$$\xi_1 + i\eta_1 = \frac{i}{w_1'},$$

in which

$$w_1 = \frac{w}{m}, \qquad w_1' = \frac{dw_1}{dz_1}, \tag{23}$$

at these values of y_1 the values of

$$w_1' = u_1 - iv_1$$

are known, with $u_1 = u\xi_0$, $v_1 = v\xi_0$. The dimensionless stream function ψ_1 can then be determined from

$$\frac{\partial \psi_1}{\partial y_1} = u_1, \tag{24}$$

by integration with respect to y_1. With ψ_1 so determined for various fixed values of x_1, the streamlines can be drawn through points with the same value of ψ_1. The flow pattern is shown in Fig. 50, in which, since $w_1 = w/m$,

$$\phi_1 = \frac{\phi}{m}, \qquad \psi_1 = \frac{\phi}{m}.$$

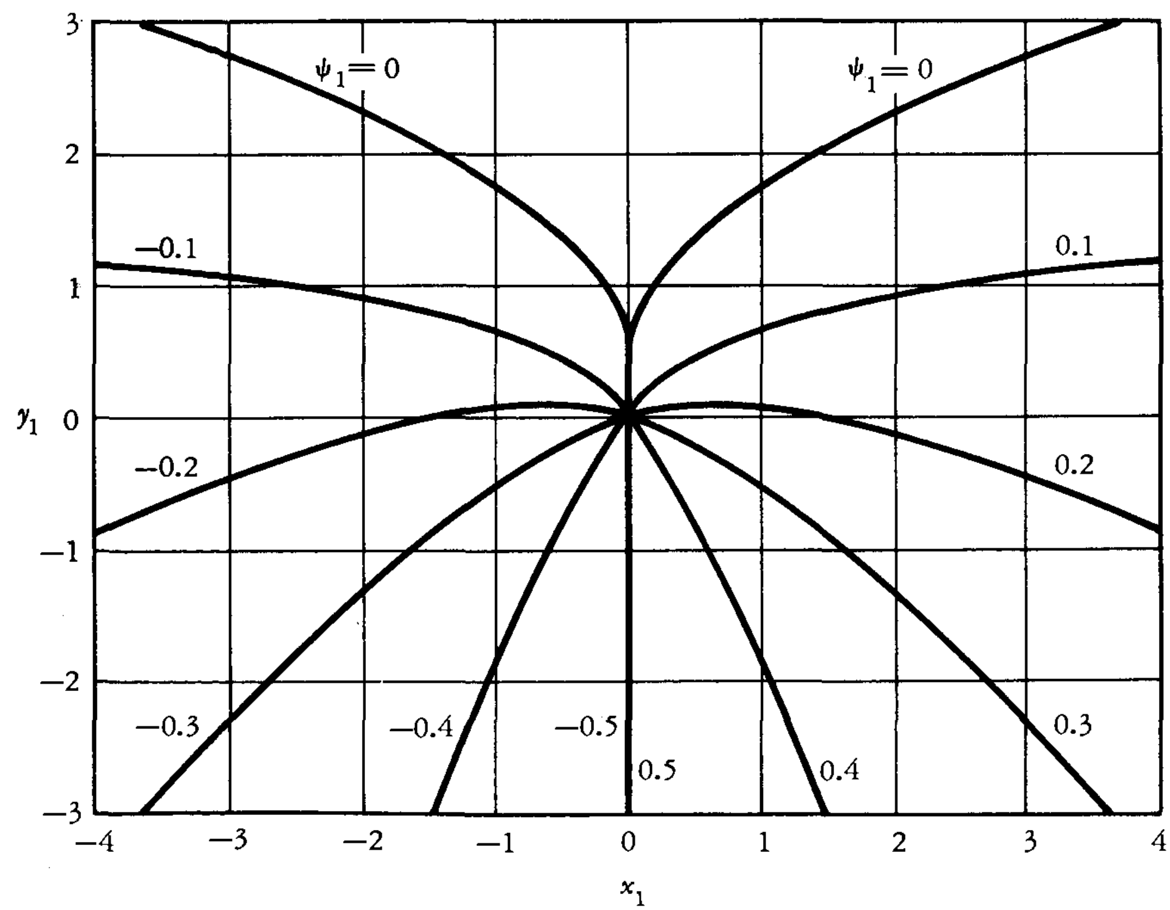


FIGURE 50. Pattern of free-surface seepage flow into a two-dimensional sink. (*Courtesy of The Physics of Fluids.*)

2.3. *Flow of a Liquid Layer Overlying a Heavier Fluid*

The problem of "water coning" is often encountered in the oil industry when a layer of water underlies a layer of oil. At sufficiently high rates of pumping, water is pumped out together with the oil in the form of a cone. Water coning must be studied as a problem of three-dimensional seepage flow. At present no solution exists giving, for a specified geometry and specified densities, the critical discharge above which the lower fluid will be pumped out. Here we shall discuss a remotely related two-dimensional problem. But even for this simplified problem the desired solution is not known. The solutions presented here serve only to give some insight into the phenomenon examined and to illustrate the lack of uniqueness of the solution.

We consider an impermeable plane separating two fluids and perforated near the center (just below the sink in Fig. 51a) to allow the lower fluid to rise above the plane.* With reference to Fig. 51a, the density of the flowing fluid (oil) is denoted by ρ_1, the density of the stagnant fluid (water) is denoted by ρ_2, and the difference $\rho_2 - \rho_1$ is denoted by $\Delta\rho$. By a development similar to that used to obtain (9), it can be shown that on the free surface BC the velocity components satisfy the condition

$$u^2 + v^2 - \frac{v}{\xi_0} = 0, \tag{25}$$

in which $1/\xi_0 = k\,\Delta\rho g/\mu$. This equation differs from (9) only in the sign preceding v/ξ_0, because the flowing fluid is now *above* the stagnant fluid. At the point B, $v = 0$. Hence at B, $u = 0$ according to (25), and B is a stagnation point. At the point C the velocity, if not zero, is in the direction of the vertical. The velocity at C will first be assumed to be different from zero. Later the case in which C is a stagnation point will be treated. It appears that the first case is the critical case, in the sense that any increase in discharge (m) will result in the appearance of water in the fluid pumped out. A more detailed discussion will be given after the solutions are obtained.

Case 1: With cusp. With ζ defined as before, the boundary of the flow region (or one half of it) is shown in Fig. 51b in the ζ-plane, and in Fig. 51d in the w-plane. Since the flow region in the ζ-plane as well as the w-plane is bounded by a polygon, the Schwarz-Christoffel transformation can be used. With the coordinates B, C, and D in the t-plane as shown in Fig. 51c, the transformation of Schwarz and Christoffel gives

$$\zeta = \frac{i\xi_0}{\pi}\cosh^{-1} t \tag{26}$$

and

$$w = \frac{m}{2\pi}[\ln(t - t_A) - \ln(t - 1)], \tag{27}$$

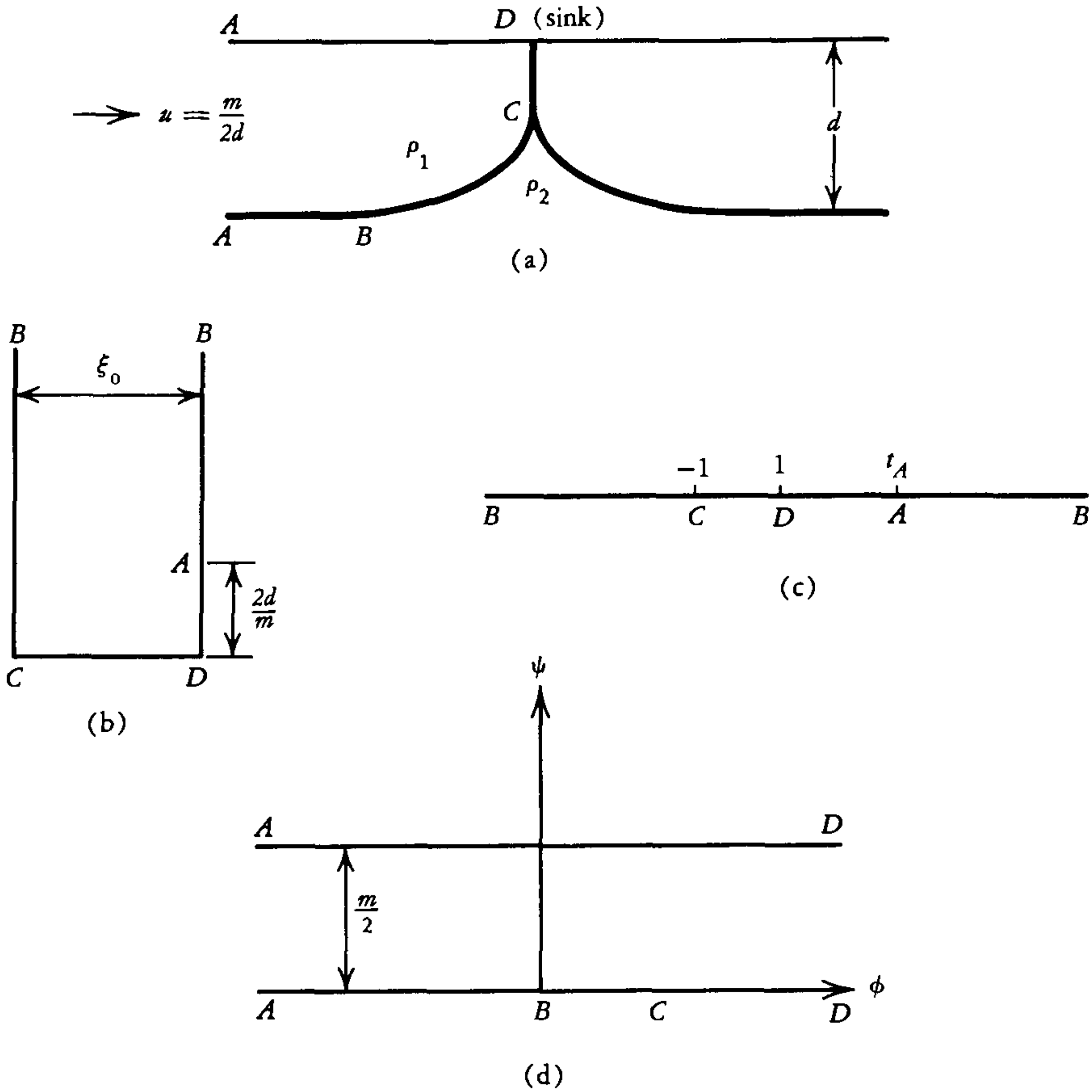


FIGURE 51. Conformal-mapping planes for Case 1 of the water-wedging problem. (a) Physical plane. (b) ζ-plane. (c) t-plane. (d) Complex-potential plane. The lower rigid boundary is along AB. (*Courtesy of The Physics of Fluids.*)

in which the constants of integration and the constant factors on the right-hand sides have been determined from the coordinates of the corner points given in the ζ-plane and w-plane. The letter m represents the discharge into the sink.

Since ζ and w are now known functions of t, and

$$\zeta = i\frac{dz}{dw},$$

z can be expressed in terms of t. In fact,

$$dz = -i\zeta\, dw = \frac{\xi_0 m}{2\pi^2}(\cosh^{-1}t)\left(\frac{1}{t - t_A} - \frac{1}{t - 1}\right) dt. \tag{28}$$

Now t is negative between B and C. Hence the imaginary part of $\cosh^{-1}t$ is πi for negative t, and integration of the imaginary part of (28) yields

$$y = \frac{\xi_0 m}{2\pi} \ln \frac{t_A - t}{1 - t} \tag{29}$$

and

$$y_c = \frac{\xi_0 m}{2\pi} \ln \frac{t_A + 1}{2}. \tag{30}$$

In (29), the constant of integration is chosen to be zero so that y_B is zero. The constant t_A is determined from the fact that $u = m/2d$ at A, so that $\zeta_A = (2d/m)i$. Thus Eq. (26) determines t_A to be $\cosh(2d\pi/m\xi_0)$, and (30) becomes

$$y_c = \frac{\xi_0 m}{2\pi}\left[\ln\left(\cosh\frac{2d\pi}{m\xi_0} + 1\right) - \ln 2\right] = \frac{\xi_0 m}{\pi} \ln \cosh \frac{d\pi}{m\xi_0}. \tag{31}$$

The shape of the free streamline is given parametrically by (28), in differential form. The expression for y is given in integrated form by (29). The expression for x can be determined numerically. The slope at B is

$$\frac{dy}{dx} = \lim_{t\to-\infty} \frac{\pi}{\cosh^{-1}|t|} = 0,$$

so that the free streamline is tangent to AB at B. This seems strange at first sight, since B is a stagnation point. But the transformations are consistent with both stagnancy and tangency at B, and the velocity at a smooth corner can be reduced to zero.

Case 2: Without cusp. In Fig. 52, where the graphs of the boundary in the four planes are shown, the point D is now assumed to be a stagnation point. Between B and D there is a point at which the speed is a maximum, designated by C. The transformation between ζ and t is

$$\zeta = M\int \frac{t - t_c}{t\sqrt{t+1}}\, dt + N = 2M(\sqrt{t+1} + t_c \tanh^{-1}\sqrt{t+1}) + N. \tag{32}$$

The constant N is zero because $\zeta_E = 0$ and $t_E = -1$. The constant M is determined by the fact that the imaginary part of $\tanh^{-1}\sqrt{t+1}$ for any positive t is $(\pi/2)i$. As t crosses B it changes from negative to positive, and ζ changes by the amount

$$Mt_c\pi i.$$

On the other hand this change is exactly $-\xi_0$. (Note that (25) has replaced (9); hence the minus sign.) Thus

$$M = \frac{i\xi_0}{t_c\pi}. \tag{33}$$

The quantity t_c is determined from the equation

$$-\xi_0 + \beta i = \frac{i2\xi_0}{\pi}\left(\frac{\sqrt{t_c + 1}}{t_c} + \tanh^{-1}\sqrt{t_c + 1}\right). \tag{34}$$

The real part of this equation is automatically satisfied because M has been determined to give the real part of ζ_c (or of ζ_B for $t = +0$) the value $-\xi_0$. The imaginary part of (34) determines t_c in terms of the parameter β, albeit implicitly:

$$\beta = \frac{2\xi_0}{\pi}\left(\frac{\sqrt{t_c + 1}}{t_c} + \coth^{-1}\sqrt{t_c + 1}\right), \tag{35}$$

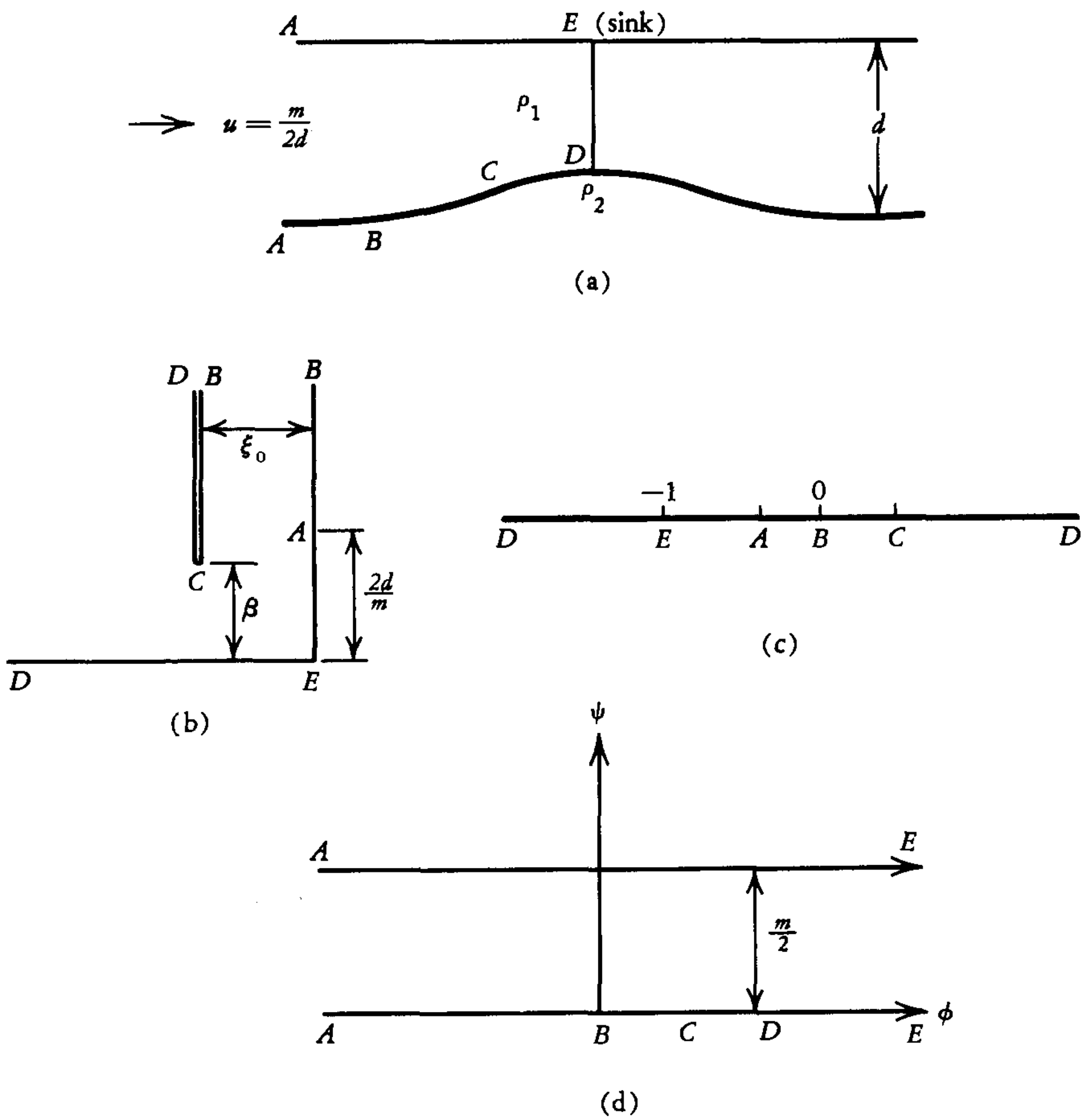


FIGURE 52. Conformal-mapping planes for Case 2 of the water-wedging problem. (a) Physical plane. (b) ζ-plane. (c) t-plane. (d) Complex-potential plane. (*Courtesy of Physics of Fluids.*)

in which

$$\coth^{-1}\sqrt{t_c + 1} = \tanh^{-1}\sqrt{t_c + 1} - \frac{\pi}{2}i.$$

Thus,

$$\zeta\left(= i\frac{dz}{dw}\right) = \frac{i2\xi_0}{\pi}\left(\frac{\sqrt{t+1}}{t_c} + \tanh^{-1}\sqrt{t+1}\right), \tag{36}$$

whereas w is still given by (27). The free streamline is then given parametrically and in differential form, by

$$dz = -i\zeta\, dw = \frac{\xi_0 m}{\pi^2}\left(\frac{\sqrt{t+1}}{t_c} + \coth^{-1}\sqrt{t+1} + \frac{\pi}{2}i\right)\left(\frac{1}{t - t_A} - \frac{1}{t+1}\right) dt. \tag{37}$$

Integration of the imaginary part of this equation yields

$$y = \frac{\xi_0 m}{2\pi}\left[\ln\frac{t - t_A}{t+1} - \ln(-t_A)\right], \tag{38}$$

the constant of integration being so determined as to satisfy the condition $y_B = 0$. In particular,

$$y_D = \frac{\xi_0 m}{2\pi}\ln\left(\frac{1}{-t_A}\right), \tag{39}$$

in which t_A is determined by

$$\frac{i2d}{m} = \frac{i2\xi_0}{\pi}\left(\frac{\sqrt{t_A + 1}}{t_c} + \tanh^{-1}\sqrt{t_A + 1}\right). \tag{40}$$

At B, dy/dx of the free streamline is again zero. Furthermore, at D the same is true, as is consistent with the assumed stagnancy at D.

We note that in Case 1 the position of B is determined by integration of Eq. (28) between $t = -1$ and $t = -\infty$, which correspond to points C and B, respectively. The result of integration of the imaginary part is already given by (31). The result for the real part then locates the position of B. The value of t_A in (28) is, as mentioned before, determined by the fact that $u = m/2d$ at A.

In Case 2 the position of B is determined by integrating (37), in which t_A, as has been said, is determined by (40). But t_c, related to β, is left free. Since (37) contains t_c the solution depends on the choice of t_c. For any choice of t_c there is a corresponding position of B for Case 2.

It is interesting that the value y_c for Case 1 is nearly equal to (but less than) d for very small m. This means that for very small discharges the interface has to rise to a high level and to descend from the center very gradually to the level of plane AB in order to have the dynamic condition on the interface maintained by the pressure drop of the flowing fluid. For very large m Eqs. (31) shows that y_c approaches zero.

For Case 2, Eqs. (39) and (40) show that for large m again $y_c = 0$, whereas as m decreases to zero t_A approaches zero and y_D approaches d from below. Thus the solutions for both cases, dealing with the flow of the upper liquid only, cannot determine the discharge at which it is possible for the lower fluid to flow into the sink. For this determination it seems, unfortunately, necessary to allow the lower fluid to flow and to seek the total discharge below which the flow of the lower fluid is dynamically impossible.

2.4. *An Inverse Method for Two-Dimensional Free-Surface Flows*

In hydrodynamics, it is sometimes profitable not to solve directly problems with given boundary conditions, but to find solutions satisfying the differential equations and to see afterwards what boundary conditions these solutions satisfy. This approach can be adopted, with the free-surface boundary condition satisfied *a priori*. The boundary other than the surface can be taken to be any streamline, so long as the region between it and the free surface is free from singularities.

For convenience the value zero will be assigned to ψ on the free surface, where ϕ is equal to $-y/\xi_0$. It is evident that the relationship

$$\frac{iz}{\xi_0} = w + i f(w) \tag{41}$$

satisfies the free-surface condition, provided $f(w)$ is real when w is real, for the real part of (41) will then be

$$\phi = -\frac{y}{\xi_0}$$

on the streamline $\psi = 0$, which represents the free surface.

For example, if $f(w) = w^2/w_0$,

$$x = \xi_0\left[\left(\frac{\phi^2 - \psi^2}{w_0}\right) + \psi\right],$$

$$y = -\xi_0\left(1 - \frac{2\psi}{w_0}\right)\phi.$$

The streamlines (ψ = constant) are the parabolas

$$x = \xi_0\left[w_0\left(\frac{\xi_0^{-1}y}{w_0 - 2\psi}\right)^2 - \frac{\psi^2}{w_0} + \psi\right], \tag{42}$$

and the potential lines (ϕ = constant) are the parabolas

$$x = \xi_0\left[\frac{\phi^2}{w_0} + \frac{w_0}{4} - \frac{w_0}{4}\left(\frac{y}{\xi_0\phi}\right)^2\right]. \tag{43}$$

With

$$x_1 = \frac{x}{w_0\xi_0}, \qquad y_1 = \frac{y}{w_0\xi_0}, \qquad \text{and} \qquad w_1 = \frac{w}{w_0} = \phi_1 + i\psi_1,$$

(42) and (43) become

$$x_1 = \left(\frac{y_1}{1 - 2\psi_1}\right)^2 - \psi_1(\psi_1 - 1) \tag{42a}$$

and

$$x_1 = \phi_1^2 + \frac{1}{4} - \frac{1}{4}\left(\frac{y_1}{\phi_1}\right)^2. \tag{43a}$$

The flow net is shown in Fig. 53. In (42a) the value of x_1 is arbitrary for $y_1 = 0$ and $\psi_1 = -\frac{1}{2}$, which means that part of the line $y_1 = 0$ is a streamline. In

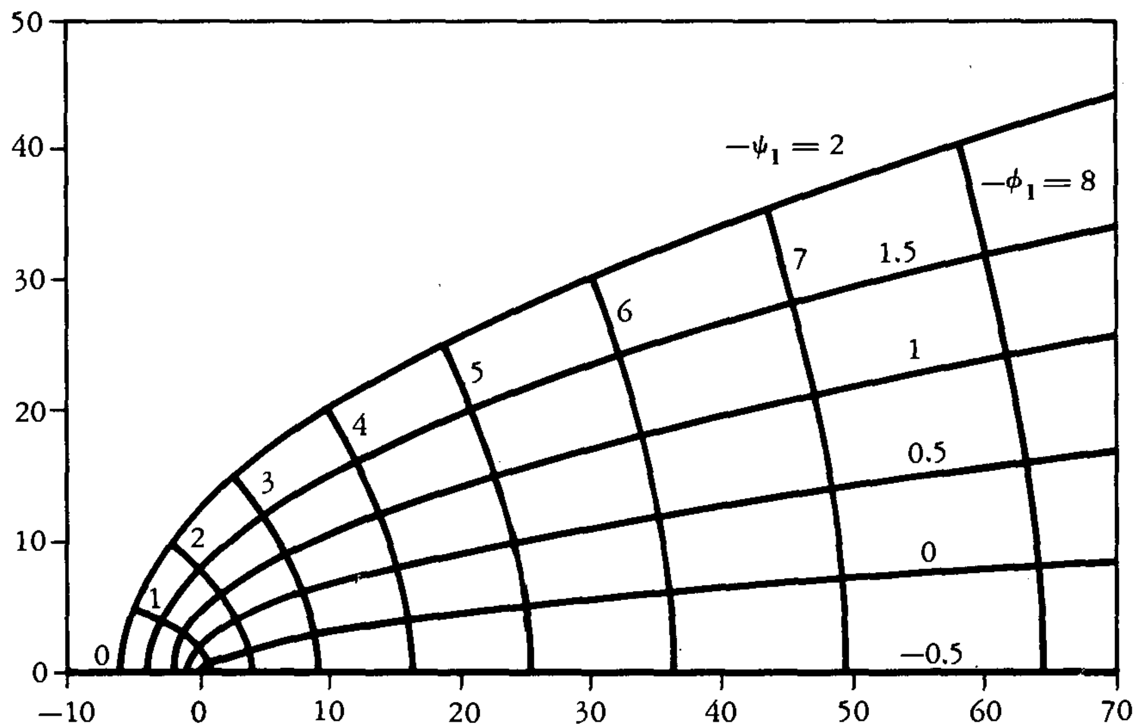


FIGURE 53. Free-surface flow into a horizontal drain.

(43a) the value of x_1 is arbitrary for $y_1 = 0$ and $\phi_1 = 0$, which means that part of the line $y_1 = 0$ is a potential line. The dividing point is $x_1 = \frac{1}{4}$, $y_1 = 0$, because any streamline with ψ_1 ever so slightly different from $-\frac{1}{2}$ will intersect the line $y_1 = 0$ very near $x_1 = \frac{1}{4}$, and similarly any potential line with ϕ_1 ever so slightly different from zero will intersect the line $y_1 = 0$ very near the same point. Thus the flow may be described as the flow from a reservoir bounded by a parabolic wall into a horizontal drain. The solution is very much the same as the one given in Muskat [1937, Chapter 6].

3. VELOCITY OF FLUID MASSES IN POROUS MEDIA

The movement of fluid masses imbedded in another fluid flowing in porous media is of interest to the oil industry because the movement of a water mass in flowing oil is a problem to be dealt with. It is also of interest to the paper industry, because in the paper-making process felts are used to carry the paper sheet through its many stages of formation and drying, and the efficiency of water removal from a *partially* saturated felt is greatly affected by the movement of air in the milieu of water flowing through the porous medium of felt. Even the answer to the apparently simple question of how much water (and how much air) will be squeezed out of a partially saturated rag put through a wringer (consisting of two cylinders) depends on the knowledge of the velocity of fluid masses in a porous medium permeated with another fluid.

The fluid mass considered in this section is either a sphere, a circular cylinder, an elliptic cylinder, or an ellipsoid. The orientation of the fluid mass with respect to either the direction of gravity or the direction of the velocity U' of the ambient fluid is entirely arbitrary.

The solution for the velocity of an ellipsoid of revolution moving in a porous medium with another fluid, in which it is immersed, has been given by Polubarinova-Kochina and Falkovich [1951]. Their solution has been presented in a more general form by Taylor and Saffman [1959]. Since the sphere and the circular cylinder are special cases of the ellipsoid of revolution, the problem for these special cases can be considered as solved. Taylor and Saffman also considered the speed of an elliptic (and a circular) bubble in a Hele-Shaw cell. Since there can be no velocity normal to the walls of the cell, the flow considered by them is necessarily two-dimensional. They obtained the solutions for a Hele-Shaw cell of finite width first, and deduced the solutions for infinite width by a limiting process. The case of the elliptic cylinder in three-dimensional motion and the case of the general ellipsoid have been treated by Yih [1963b]. Although the results for a sphere, a circular cylinder, and an elliptic cylinder can be deduced from that for the general ellipsoid, the four cases will be presented separately. This procedure is adopted because it is not always easy to deduce a special result from a general one, and because self-contained derivations for simple and special cases may actually be preferred by some readers.

3.1. *Velocity of a Fluid Mass in the Form of a Circular Cylinder*

The direction of the velocity U' (at infinity) of the ambient fluid will be taken to be the x'-direction. The axis of the fluid cylinder will be taken as the x-axis.* The y-axis and z-axis lie in a plane perpendicular to the x-axis, are normal to each other, but are otherwise arbitrarily oriented. The direction

* The coordinates are fixed in space. The orientation of the fluid mass with respect to these fixed coordinates is instantaneous.

cosines of the x'-axis with respect to the x-y-z coordinates will be denoted by α', β', and γ'. The direction cosines of the Z-axis will be denoted by α, β, and γ. The components of U' in the directions of increasing x, y, and z are $\alpha'U'$, $\beta'U'$, and $\gamma'U'$, respectively. For convenience these will be denoted by U_1, V_1, and W_1.

The first of Eqs. (1) is

$$\frac{\mu}{k} u = -\frac{\partial p}{\partial x} - \rho g \alpha. \tag{44}$$

Since the u for the ambient fluid is everywhere U_1, for that fluid

$$-\frac{\partial p}{\partial x} = \frac{\mu_1}{k} U_1 + \rho_1 g \alpha, \tag{45}$$

in which the subscript 1 indicates the ambient fluid. The fluid mass will be assumed to move as a solid body. This assumption will be verified *a posteriori*. Thus, the velocity components of the fluid mass are constant, and will be denoted by U_2, V_2, and W_2. For the fluid mass, (44) takes the form

$$-\frac{\partial p}{\partial x} = \frac{\mu_2}{k} U_2 + \rho_2 g \alpha, \tag{46}$$

in which the subscript 2 refers to the fluid mass. If the pressure is to be the same at the boundary of the two fluids, the left-hand sides of (45) and (46) must be the same, and

$$U_2 = \frac{\mu_1}{\mu_2} U_1 + \frac{k(\rho_1 - \rho_2) g \alpha}{\mu_2}. \tag{47}$$

In fact, this relation holds for the axial component of the velocity of a cylinder of whatever cross section. If $(\rho_1 - \rho_2)\alpha = 0$,

$$U_2 = \frac{\mu_1}{\mu_2} U_1. \tag{48}$$

For motion of the fluid cylinder in the y-direction, the differential equation governing the flow of the ambient fluid is

$$\frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0,$$

in which

$$\phi_1 = -\frac{k}{\mu_1}(p + \rho_1 g \beta y) \tag{49}$$

is that part of the potential which is due to the motion of the cylinder in the y-direction alone. Since the fluid cylinder is assumed to move like a solid body, the potential ϕ_1 for the components V_1 and V_2 is

$$\phi_1 = V_1 y - (V_2 - V_1) a^2 y (y^2 + z^2)^{-1}, \tag{50}$$

in which a is the radius of the cylinder. This solution is well known. The first term on the right-hand side of (50) represents a uniform flow in the y-direction with speed V_1, and the second term represents the potential due to a doublet at the origin, with its axis in the y-direction and the appropriate strength to satisfy the kinematic condition at the interface that the velocity components normal to it must be the same for both fluids. As for the fluid cylinder, since V_2 is constant, the potential is simply

$$\phi_2 = V_2 y. \tag{51}$$

Since

$$\phi_2 = -\frac{k}{\mu_2}(p + \rho_2 g\beta y), \tag{52}$$

and the pressure p in (49) and (52) must be the same on the interface, equations (49) to (52) give

$$\frac{\mu_1}{k}[-V_1 y + (V_2 - V_1)y] - \rho_1 g\beta y = -\frac{\mu_2}{k}V_2 y - \rho_2 g\beta y,$$

or

$$V_2 = \frac{2\mu_1}{\mu_1 + \mu_2}V_1 + \frac{k(\rho_1 - \rho_2)}{\mu_1 + \mu_2}g\beta. \tag{53}$$

For $\mu_2 = 0$ and $(\rho_1 - \rho_2)\beta = 0$, $V_2 = 2V_1$, in agreement with the result of Taylor and Saffman. The development for the z-direction is entirely similar, and the result corresponding to (53) is

$$W_2 = \frac{2\mu_1}{\mu_1 + \mu_2}W_1 + \frac{k(\rho_1 - \rho_2)}{\mu_1 + \mu_2}g\gamma. \tag{54}$$

The development hitherto shows that the solid-body motion assumed for the cylinder is dynamically possible.

Equations (47), (53), and (54) indicate that the velocity of the fluid mass is entirely independent of size, and that for an ambiently quiescent fluid ($U' = 0$) the velocity of the cylinder is in the direction of gravity if $\rho_2 > \rho_1$, and in the opposite direction if $\rho_1 > \rho_2$, the speed being

$$\frac{k|\rho_2 - \rho_1|g}{\mu_2 + \mu_1}$$

in either case.

3.2. *Velocity of a Cylindrical Mass of Elliptic Cross Section*

The axis of the cylinder is again chosen to be the x-axis. The elliptic cross section of the cylinder is described by (Fig. 54)

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1, \tag{55}$$

which fixes the y- and z-axes. For the x-component of the motion (47) is again valid. For the y- and z-directions, it is best to use the transformation

$$y + iz = c \cosh(\xi + i\eta),$$

or

$$y = c \cosh \xi \cos \eta, \qquad z = c \sinh \xi \sin \eta. \tag{56}$$

On the ellipse, $\xi = \xi_0$, because (55) and (56) coincide if

$$a = c \cosh \xi_0, \qquad b = c \sinh \xi_0,$$

which defines c and ξ_0 in terms of a and b. The complex potential for the flow caused by a velocity ΔV in the y-direction relative to the ambient fluid is

$$\phi + i\psi = Ce^{-(\xi + i\eta)}, \tag{57}$$

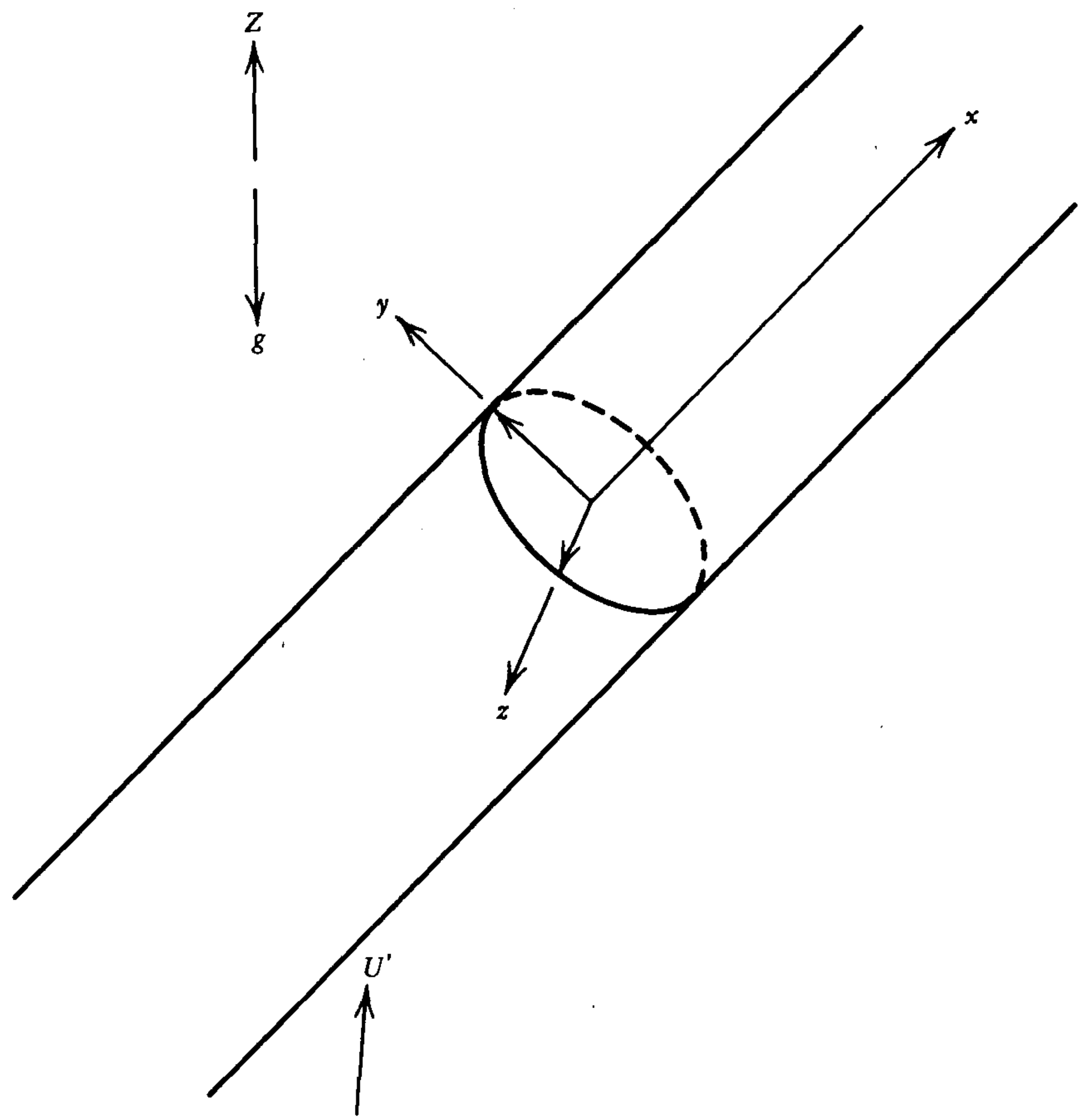


FIGURE 54. Definition sketch for the case of an elliptic fluid cylinder moving in another fluid with ambient velocity U'. The direction cosines of gravity are α, β, and γ. Those of U' are α', β', and γ'. (*Courtesy of The Physics of Fluids.*)

The boundary condition on the ellipse is

$$\psi = \Delta V\, z + \text{constant},$$

which is satisfied if

$$Ce^{-\xi_0} = -\Delta V\, c \sinh \xi_0 ,$$

since

$$\psi = -Ce^{-\xi} \sin \eta.$$

Thus

$$C = -\frac{\Delta V\, bc}{a - b} = -\Delta V\, b \left(\frac{a + b}{a - b}\right)^{1/2}$$

and

$$\phi = Ce^{-\xi} \cos \eta.$$

On the ellipse

$$\phi = Ce^{-\xi_0} \cos \eta = -\Delta V\, (\tanh \xi_0) y = -\frac{b}{a}\, \Delta V\, y.$$

Similarly, if the velocity component in the z-direction of the cylinder relative to the ambient fluid is ΔW, that part of the complex potential due to ΔW is

$$\phi + i\psi = iDe^{-(\xi + i\eta)},$$

with

$$D = -\Delta W\, a \left(\frac{a + b}{a - b}\right)^{1/2}$$

and

$$\phi = -\frac{a}{b}\, \Delta W\, z.$$

On the ellipse, with $\Delta V = V_2 - V_1$,

$$\phi_1 = V_1 y - \frac{(V_2 - V_1) by}{a},$$

$$\phi_2 = V_2 y,$$

if the elliptic cylinder is assumed to move as a solid body. Continuity of p then demands that

$$\frac{\mu_1}{k}\left[-V_1 y + \frac{(V_2 - V_1) by}{a}\right] - \rho_1 g \beta y = -\frac{\mu_2}{k}\, V_2 y - \rho_2 g \beta y,$$

which gives

$$V_2 = \frac{\mu_1(a + b)}{\mu_1 b + \mu_2 a}\, V_1 + \frac{ak(\rho_1 - \rho_2) g \beta}{\mu_1 b + \mu_2 a}. \tag{58}$$

Similarly,

$$W_2 = \frac{\mu_1(a + b)}{\mu_1 a + u_2 b}\, W_1 + \frac{bk(\rho_1 - \rho_2) g \gamma}{\mu_1 a + \mu_2 b}. \tag{59}$$

From (47), (58), and (59) it can be seen that the velocity of the cylinder is dependent on its shape, but not on its size.

If $\rho_1 = \rho_2$,

$$\frac{W_2}{V_2} = \frac{\mu_1 b + \mu_2 a}{\mu_1 a + \mu_2 b}\frac{W_1}{V_1} = \left[1 + \frac{(a - b)(\mu_2 - \mu_1)}{\mu_1 a + \mu_2 b}\right]\frac{W_1}{V_1}.$$

If, in addition, $\mu_2 < \mu_1$, then since $a > b$,

$$\frac{W_2}{V_2} < \frac{W_1}{V_1},$$

so that in the y–z plane the velocity of the cylinder will not be in the same direction as that of the ambient fluid, but will deviate from it toward the major axis of the ellipse. The reverse is true if $\mu_2 > \mu_1$. The fluid mass chooses to move in a direction of less resistance.

If $U' = 0$,

$$U_2 : V_2 : W_2 = (\mu_1 b + \mu_2 a)\alpha : \mu_2 a\beta : \mu_2 b\gamma,$$

so that the velocity of the cylinder will not be in the direction of gravity, but will deviate from it toward the y-axis, and even more toward the x-axis. In other words, it will drift in such a way as to favor the axes of the cylinder in the order of their lengths: ∞, a, and b.

3.3. *Velocity of a Spherical Mass*

In the case of a spherical mass the x-axis can be arbitrarily chosen. The potential for the x-component of the motion of the sphere is

$$\phi_1 = U_1 x - \frac{(U_2 - U_1)a^3 x}{2R^3},$$

in which a is the radius of the sphere, and R is the radial distance from the sphere. On the sphere this becomes

$$\phi_1 = \left[U_1 - \frac{U_2 - U_1}{2}\right]x.$$

On the other hand, if the sphere moves as a solid body, the potential for fluid 2 for the x-component of the motion is

$$\phi_2 = U_2 x.$$

Continuity of pressure then demands

$$\frac{\mu_1}{2k}(U_2 - 3U_1)x - g\rho_1\alpha x = -\frac{\mu_2}{k}U_2 x - g\rho_2\alpha x,$$

or

$$U_2 = \frac{3\mu_1}{2\mu_2 + \mu_1}U_1 + \frac{2kg(\rho_1 - \rho_2)\alpha}{2\mu_2 + \mu_1}. \tag{60a}$$

Similarly,

$$V_2 = \frac{3\mu_1}{2\mu_2 + \mu_1} V_1 + \frac{2kg(\rho_1 - \rho_2)\beta}{2\mu_2 + \mu_1}, \tag{60b}$$

$$W_2 = \frac{3\mu_1}{2\mu_2 + \mu_1} W_1 + \frac{2kg(\rho_1 - \rho_2)\gamma}{2\mu_2 + \mu_1}. \tag{60c}$$

If $\mu_2 = 0$ and $\rho_1 = \rho_2$, $U_2 = 3U_1$, in agreement with Taylor and Saffman's result derived from the solution for an ellipsoid of revolution. If $U_1 = 0$, the sphere moves in the direction of gravity or in a direction opposite to it, according as $\rho_2 > \rho_1$ or $\rho_2 < \rho_1$. In either case the speed is

$$\frac{2kg|\rho_1 - \rho_2|}{2\mu_2 + \mu_1}.$$

3.4. *Velocity of an Ellipsoidal Mass*

If the axes of the ellipsoid are taken to be the coordinate axes, its surface is described by (Fig. 55)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{61}$$

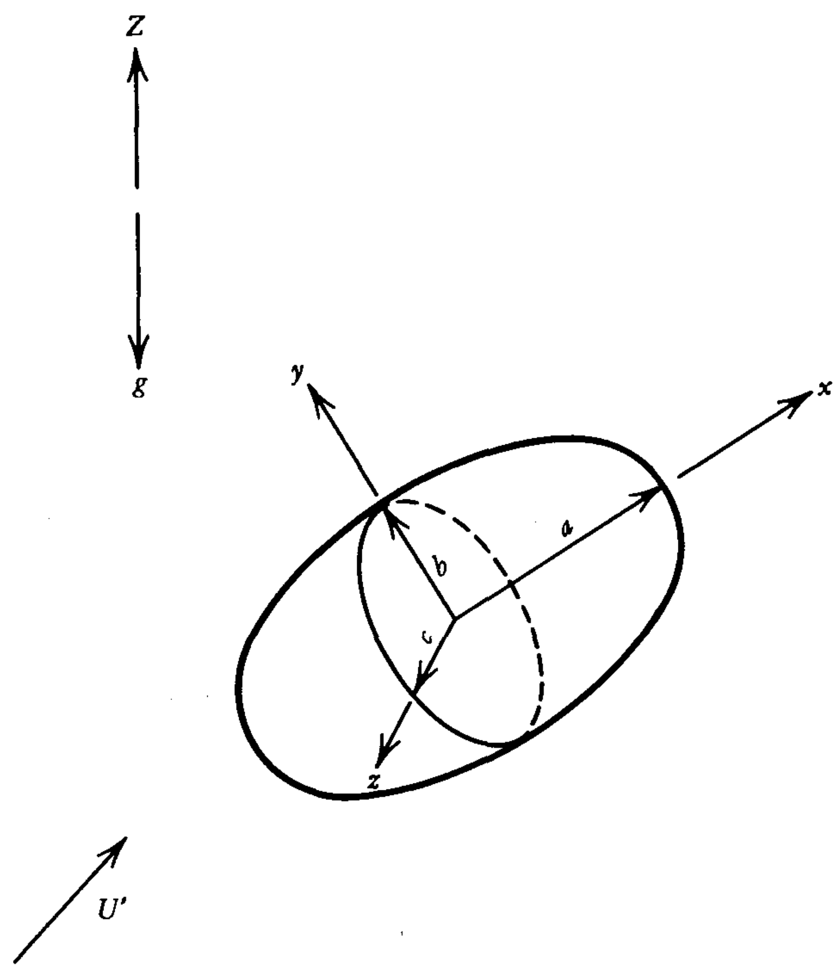


FIGURE 55. Definition sketch for the case of an ellipsoidal mass of fluid moving in another fluid of ambient velocity U'. (*Courtesy of The Physics of Fluids.*)

in which a, b, and c are semiaxes. The potentials for velocity components ΔU, ΔV, and ΔW of the ellipsoidal mass relative to the ambient fluid are, respectively,

$$\phi = -\frac{abc}{2-\alpha_0}\,\Delta U\,x\int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)\Delta}, \tag{62}$$

$$\phi = -\frac{abc}{2-\beta_0}\,\Delta V\,y\int_\lambda^\infty \frac{d\lambda}{(b^2+\lambda)\Delta}, \tag{63}$$

and

$$\phi = -\frac{abc}{2-\gamma_0}\,\Delta W\,z\int_\lambda^\infty \frac{d\lambda}{(c^2+\lambda)\Delta}, \tag{64}$$

in which

$$\alpha_0 = abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)\Delta}, \qquad \beta_0 = abc\int_0^\infty \frac{d\lambda}{(b^2+\lambda)\Delta},$$
$$\gamma_0 = abc\int_0^\infty \frac{d\lambda}{(c^2+\lambda)\Delta}, \tag{65}$$

and

$$\Delta = [(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]^{1/2}.$$

The coordinate λ is the first of the ellipsoidal coordinates λ, μ, and ν, which are roots of the cubic equation in θ,

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1,$$

for every point (x, y, z). On the surface of the ellipsoid given by (60), $\lambda = 0$, and

$$\phi = -\frac{\alpha}{2-\alpha_0}\,\Delta U\,x, \qquad \phi = -\frac{\beta_0}{2-\beta_0}\,\Delta V\,y, \qquad \phi = -\frac{\gamma_0}{2-\gamma_0}\,\Delta W\,z$$

for the three modes of motion. For the motion parallel to the x-axis,

$$\phi_1 = U_1 x - \frac{\alpha_0}{2-\alpha_0}(U_2 - U_1)x$$

for the ambient fluid, and

$$\phi_2 = U_2 x$$

for the ellipsoidal fluid mass, which is assumed in solid-body translation. The continuity of pressure at the surface of the ellipsoid demands that

$$\frac{\mu_1}{k}\left[-U_1 x + \frac{\alpha_0}{2-\alpha_0}(U_2-U_1)x\right] - g\rho_1\alpha x = -\frac{\mu_2}{k}U_2 x - g\rho_2\alpha x,$$

from which

$$U_2 = \frac{2\mu_1 U_1 + (2-\alpha_0)kg(\rho_1-\rho_2)\alpha}{\alpha_0\mu_1 + (2-\alpha_0)\mu_2}. \tag{66a}$$

Similarly,

$$V_2 = \frac{2\mu_1 V_1 + (2 - \beta_0)kg(\rho_1 - \rho_2)\beta}{\beta_0\mu_1 + (2 - \beta_0)\mu_2} \tag{66b}$$

and

$$W_2 = \frac{2\mu_1 W_1 + (2 - \gamma_0)kg(\rho_1 - \rho_2)\gamma}{\gamma_0\mu_1 + (2 - \gamma_0)\mu_2}. \tag{66c}$$

For a sphere $a = b = c$, and

$$\alpha_0 = \beta_0 = \gamma_0 = \tfrac{2}{3}.$$

Thus Eqs. (66) reduce to (60), providing an independent check.

For the case $\rho_1 = \rho_2$,

$$\frac{U_2}{U_1} > \frac{V_2}{V_1} > \frac{W_2}{W_1}$$

if $\mu_2 < \mu_1$, and

$$\frac{U_2}{U_1} < \frac{V_2}{V_1} < \frac{W_2}{W_1}$$

if $\mu_2 > \mu_1$, since $\alpha_0 < \beta_0 < \gamma_0$. This means that the ellipsoid chooses to move in a direction of less resistance. For the case $U' = 0$,

$$U_2 : V_2 : W_2$$

$$= \frac{\alpha}{[\alpha_0/(2 - \alpha_0)]\mu_1 + \mu_2} : \frac{\beta}{[\beta_0/(2 - \beta_0)]\mu_1 + \mu_2} : \frac{\gamma}{[\gamma_0/(2 - \gamma_0)]\mu_1 + \mu_2}.$$

Since $\alpha_0 < \beta_0 < \gamma_0$, the direction of the ellipsoid will deviate from that of gravity toward the direction of y and even more toward the direction of x, whether μ_2 is greater or less than μ_1. In other words, the ellipsoid will drift in such a way as to favor its axes in the order of their lengths. The independence of the sign of $\mu_1 - \mu_2$ is due to the fact that the inequality of the three axes is not merely a measure of the inequality of the resistances to flow in the three directions, as in the case $\rho_1 = \rho_2$, but is also a measure of the inequality of the three components of the motive force in the three directions, when $\rho_1 \neq \rho_2$. The latter inequality dominates the flow when $U' = 0$, since the flow is then entirely motivated by gravity.

4. STEADY FLOWS OF A NONHOMOGENEOUS FLUID IN POROUS MEDIA

It has been shown that the flow of a homogeneous incompressible fluid through porous media is governed by the Laplace equation. If the fluid is not homogeneous, the governing equation or equations are much more complicated. In general, the three equations in (1) have to be used instead of the Laplace equation. However, if the flow is steady, the effect of viscosity variation (from streamline to streamline) can be determined once and for all, for general three-dimensional flows, provided diffusion is neglected. Furthermore, a single equation governing two-dimensional or axisymmetric flows can be found even if there is density variation and the attending gravity effect is taken into account [Yih, 1961a]. As shown by Saffman [1959 and 1960], diffusion in flows through porous media can be neglected if the scale of the flow is large compared with the dimension of the interstices of the medium. This condition is met not only in nature but also in most laboratory models.

4.1. *Effect of Viscosity Variation*

In the absence of gravity or of density variation, the effect of viscosity variation on steady flows can be ascertained in a definite and simple manner for general three-dimensional flows. Since the case of constant ρ (which also implies the absence of a free surface) and the case of zero gravity are quite the same, because the only effect of gravity in the former case is to change the pressure by an amount according to the law of hydrostatics, one may consider the case of no body force, for simplicity. If the flow is steady, (1) becomes

$$\frac{\mu}{k} u_i = -\frac{\partial p}{\partial x_i}, \tag{67}$$

in which k is constant but μ, varying from streamline to streamline, remains constant on a streamline, so that

$$u_\alpha \frac{\partial \mu}{\partial x_\alpha} = 0. \tag{68}$$

If the velocity u_i' of an associated flow is defined by [Yih, 1961a]

$$u_i' = \frac{\mu}{\mu_0} u_i, \tag{69}$$

in which μ_0 is a reference viscosity, (67) and the equation of continuity can be written, in virtue of (68), as

$$\frac{\mu_0}{k} u_i' = -\frac{\partial p}{\partial x_i} \tag{70}$$

and

$$\frac{\partial u_i'}{\partial x_i} = 0. \tag{71}$$

But (70) and (71) are precisely the equations governing the flow of a homogeneous liquid, and can therefore be reduced to the Laplace equation. One can, therefore, solve the Laplace equation in the usual manner. When the solution for u_i' is obtained, the actual velocity field is determined by (69), provided μ is given at some section (and therefore along all the streamlines). The pressure is the same as in the associated flow. Thus, the effect of viscosity is to reduce the local velocity by a factor inversely proportional to μ.

As in the case of steady flows of a fluid of variable density or entropy, the success of the transformation (69) depends on (68), which is an equation of particular history. It was, in fact, suggested by the transformations (18) and (27) of Chapter 1.

The clear-cut effect of viscosity variation, demonstrated for the simple case of no variation in specific weight, is nevertheless still there when the specific weight varies. This can be seen by considering two steady flows having identical streamline patterns and nonuniform density distributions in a gravitational field, but with different viscosity distributions. Such flows are dynamically possible. If μ_1 is the viscosity along a streamline of the first flow, and μ_2 that on the corresponding streamline of the second, then the velocities at corresponding points on these streamlines have the ratio $C\mu_2/\mu_1$, in which C is a constant for the *entire* field of flow, and is in fact the ratio k_1/k_2, provided the permeabilities are constant over the whole field.

4.2. *Equation Governing Two-Dimensional Flows in a Gravitational Field*

Since acceleration is very small in seepage flows, density variation has no inertia effect at all. In other words, in the absence of gravity, density variation does not affect seepage flow in any way. If the density varies from place to place (or from streamline to streamline in steady flows) in a gravitational field, general three-dimensional flows can only be determined by solving (1). However, if the flow is steady and two-dimensional or axisymmetric, a single governing equation can be found for each case. Henceforth until the end of Section 5, the direction of decreasing z will be taken to be that of the gravitational acceleration, unless otherwise stated.

For two-dimensional flows, (1) can be written

$$\frac{\mu_0}{k} u' = -\frac{\partial p}{\partial x}, \qquad \frac{\mu_0}{k} w' = -\frac{\partial p}{\partial z} - g\rho, \tag{72}$$

in which $u' = u'_1$, $w' = u'_3$, and ρ is a function of the stream function ψ' only, because ρ does not change along a path line, which is a streamline in steady flow. Since

$$u' = \frac{\partial \psi'}{\partial z}, \qquad w' = -\frac{\partial \psi'}{\partial x}, \tag{73}$$

cross differentiation of (72) produces

$$\nabla^2 \psi' = \frac{kg}{\mu_0} \frac{d\rho}{d\psi'} \frac{\partial \psi'}{\partial x}, \tag{74}$$

which is the equation sought.

4.3. *Equation Governing Axisymmetric Flows in a Gravitational Field*

For axisymmetric flows, the equations of motion are

$$\frac{\mu_0}{k} u' = -\frac{\partial p}{\partial r}, \qquad \frac{\mu_0}{k} w' = -\frac{\partial p}{\partial z} - g\rho, \tag{75}$$

with u' now denoting the modified radial velocity. In terms of Stokes' stream function ψ', the velocity components are

$$u' = -\frac{1}{r}\frac{\partial \psi'}{\partial z}, \qquad w' = \frac{1}{r}\frac{\partial \psi'}{\partial r}. \tag{76}$$

Elimination of p from (75) produces, after multiplication by r,

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi' = -\frac{kgr}{\mu_0}\frac{d\rho}{d\psi'}\frac{\partial \psi'}{\partial r}, \tag{77}$$

which is the equation governing axisymmetric flow. If gravity is acting toward the center of a sphere, a similar equation governing axisymmetric flows in spherical coordinates can be derived. This will not be done because in the investigation of seepage flow it is seldom necessary to consider scales large enough to call for the use of spherical coordinates.

4.4. *Exact Solutions for Two-Dimensional Flows*

Although (74) is in general nonlinear, for a special upstream condition $d\rho/d\psi'$ is a constant, and (74) is linear. Thus, if the flow is confined between two horizontal impermeable boundaries, the flow at infinity is horizontal, and, whatever the density distribution is upstream, $d\psi'/\partial x$ is zero, and (74) becomes, at $x = -\infty$,

$$\frac{d^2\psi'}{dz^2} = 0,$$

the solution of which is

$$\psi' = U'z \qquad (at \quad x = -\infty).$$

The constant U' is just the uniform velocity of the *associated* flow. Now if the density distribution far upstream is

$$\rho = \rho_0 - \frac{\rho_0 - \rho_1}{d} z,$$

in which ρ_0 is the density at the lower boundary ($z = 0$) and ρ_1 the density at the upper boundary ($z = d$),

$$\frac{d\rho}{d\psi'} = -\frac{\rho_0 - \rho_1}{U'd}. \tag{78}$$

This relationship, determined at a section far upstream, is valid for the entire field of flow which originates from that section. With the substitutions

$$\xi = \frac{x}{d}, \qquad \eta = \frac{z}{d}, \qquad \Psi = \frac{\psi'}{U'd}, \qquad B = \frac{kg(\rho_0 - \rho_1)}{\mu_0 U'},$$

(74) assumes the dimensionless form

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\Psi + B\frac{\partial\Psi}{\partial\xi} = 0. \tag{79}$$

If the flow is caused by a sink situated at the upper boundary, of strength $2U'd$ for the *associated* flow, only the region $0 \le z \le d$, $-\infty < x \le 0$ need be considered, because of symmetry. The boundary conditions are

$$\begin{aligned}
\Psi &= 0 \quad &&\text{at} \quad \eta = 0,\\
\Psi &= 0 \quad &&\text{at} \quad \xi = 0 \quad \text{for} \quad \eta < 1,\\
\Psi &= 1 \quad &&\text{at} \quad \eta = 1,\\
\Psi &= \eta \quad &&\text{at} \quad \xi = -\infty.
\end{aligned}$$

The differential system consisting of these boundary conditions and the differential equation (79) can be solved by the method of separation of variables. The solution is

$$\Psi = \eta + \frac{2}{\pi}\sum_{n+1}^{\infty}\frac{(-1)^n}{n}\sin n\pi\eta\, e^{\alpha_n\xi}, \tag{80}$$

with

$$\alpha_n = \tfrac{1}{2}(-B + \sqrt{B^2 + 4n^2\pi}\,).$$

The flow pattern for $B = 0$ is the same pattern as for irrotational flow into a sink, and is shown in Fig. 9a of Chapter 3. Those for $B = \pi$, 2π, and 4π are shown in Figs. 56a, 56b, and 56c, respectively. It can be seen that although the flow condition at $\xi = -\infty$ is the same for all cases, the streamlines become more and more concentrated near the upper boundary as the sink is approached.

If the flow is caused by a pressure gradient in the x-direction and there is one or more impermeable ridges in the way, the solution of the problem can be obtained by two inverse methods. For either method the solution is of the form

$$\Psi_- = \eta + \sum_{n=1}^{\infty} A_n \sin n\pi\eta\, e^{\alpha_n\xi} \qquad (\xi < 0), \tag{81}$$

$$\Psi_+ = \eta + \sum_{n=1}^{\infty} B_n \sin n\pi\eta\, e^{\beta_n\xi} \qquad (\xi > 0), \tag{82}$$

in which

$$(\alpha_n, \beta_n) = \tfrac{1}{2}(-B \pm \sqrt{B^2 + 4n^2\pi^2}),$$

and B is again assumed to be constant. The first method is the method of vortex distributions, and consists in matching Ψ_- to Ψ_+ at $\xi = 0$ through the conditions

$$\Psi_- = \Psi_+ \quad \text{at} \quad \xi = 0, \tag{83}$$

$$\frac{\partial \Psi_-}{\partial \xi} - \frac{\partial \Psi_+}{\partial \xi} = f(\eta) \quad \text{at} \quad \xi = 0, \tag{84}$$

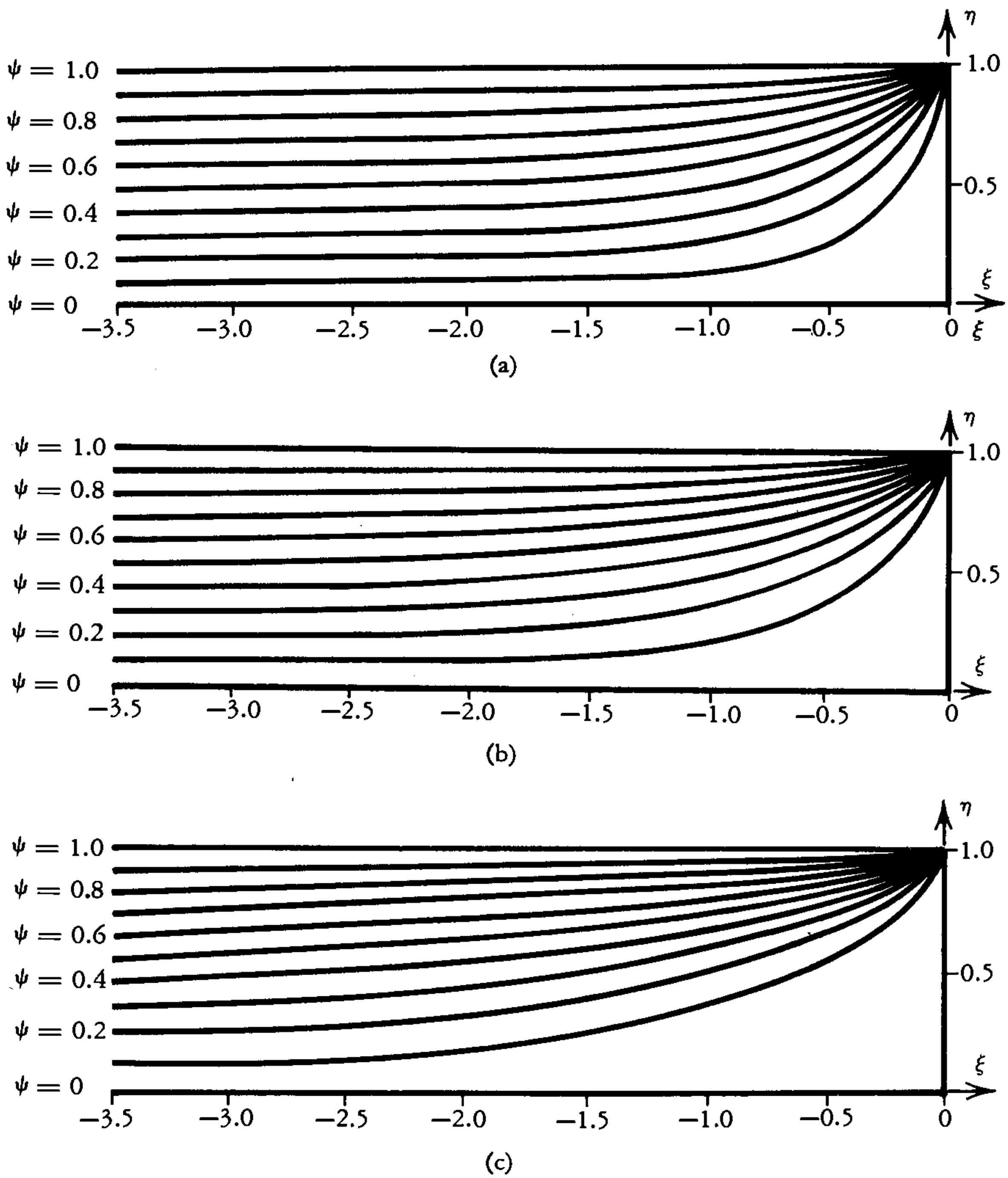


FIGURE 56. Flow of a stratified fluid through a layer of porous medium into a two-dimensional sink. (a) $B = \pi$. (b) $B = 2\pi$. (c) $B = 4\pi$. (*J. Fluid Mech., 10, part 1. Courtesy of the Cambridge Univ. Press.*)

in which $f(\eta) = 0$ for $a < \eta \leq 1$, but is otherwise arbitrary. The function $f(\eta)$, extending from $z = 0$ to $z = a$, is the vortex-distribution function generating the barrier. Equations (83) and (84) demand, respectively, that

$$A_n = B_n$$

and

$$\alpha_n A_n - \beta_n B_n = 2 \int_0^1 f(\eta) \sin n\pi\eta \, d\eta,$$

or

$$\sqrt{B^2 + 4n^2\pi^2}\, A_n = 2 \int_0^1 f(\eta) \sin n\pi\eta \, d\eta.$$

The vortex distribution does not have to be located at $\xi = 0$. If it is located at $\xi = b$, the same procedure can be followed after ξ is replaced by $\xi - b$. And, indeed, several such distributions can be used, with the solutions matched at each location. Furthermore, a concentrated vortex can be represented by a $f(\eta)$ in the form of a "Dirac function." After the solution has been obtained, the streamline $\Psi = 0$ can be traced out. This is the lower boundary. If it is not what has been specified, changes in the strengths, extent, and locations of the vortex distributions can bring it arbitrarily near the specified (lower) boundary. The vortex distribution

$$f(\eta) = \begin{cases} -12 \sin 2\pi\eta & \text{in } \ 0 \leq \eta \leq \tfrac{1}{2}, \\ 0 & \text{elsewhere} \end{cases}$$

generates flow patterns shown in Figs. 57a and 57b, for $B = \pi$ and 2π respectively.

The second method is a method of sources and sinks, in which the conditions

$$\frac{\partial \Psi_-}{\partial \xi} = \frac{\partial \Psi_+}{\partial \xi} \tag{85}$$

and

$$\Psi_- - \Psi_+ = f(\eta) \tag{86}$$

are imposed at $\xi = 0$. The function $f(\eta)$ now represents a source-sink distribution at $\xi = 0$, and is zero except in $0 \leq \eta \leq a$. Equations (85) and (86) demand that

$$\alpha_n A_n - \beta_n B_n = 0$$

and

$$\left(1 - \frac{\alpha_n}{\beta_n}\right) A_n = \left(2 - \frac{B}{n^2\pi^2}\, \alpha_n\right) A_n = 2 \int_0^1 f(\eta) \sin n\pi\eta \, d\eta.$$

The source-sink distribution can again be located at more than one value of ξ, and concentrated sources and sinks can again be represented by $f(\eta)$ in the form of "Dirac functions." But the total algebraic sum of all sources and sinks must be zero if the boundary streamline $\Psi = 0$ is to return to $\eta = 0$ after the barrier or barriers. This sum is

$$\sum_{n=1}^{N} \int_{0}^{a_n} f(\eta)\, d\eta,$$

summed over the N values of ξ, at which the singularities are located.

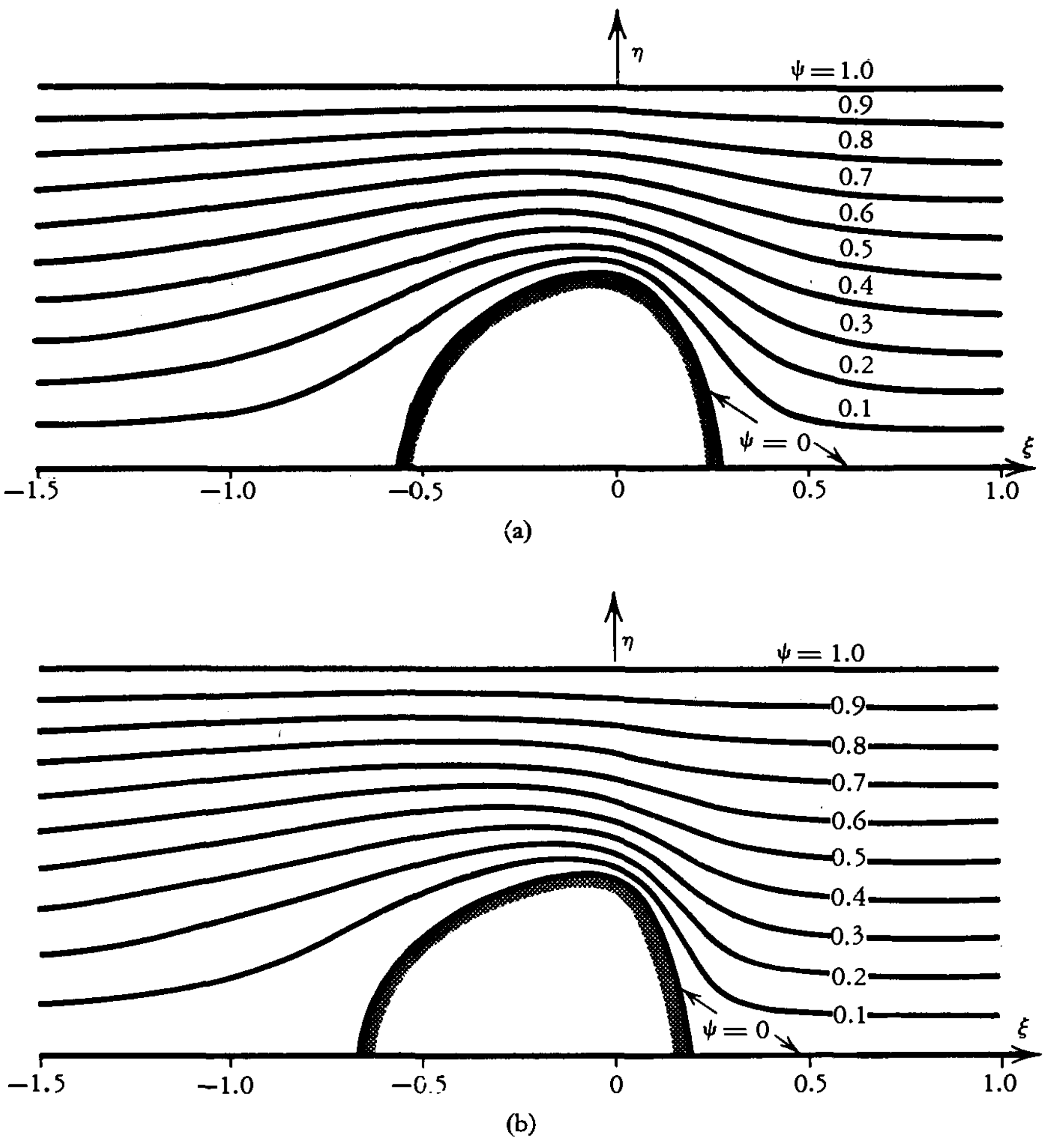


FIGURE 57. Two-dimensional flow of a stratified fluid through a layer of porous medium over an impermeable barrier. (a) $B = \pi$. (b) $B = 2\pi$. (*J. Fluid Mech.*, *10, part 1. Courtesy of the Cambridge Univ. Press.*)

5. STRATIFIED FLOW IN HELE-SHAW CELLS

Fluid flow confined between two narrowly spaced rigid boundaries can be described by

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}, \tag{87}$$

$$\mu \frac{\partial^2 w}{\partial y^2} = \frac{\partial p}{\partial z} + g\rho, \tag{88}$$

if the flow is steady and if the Reynolds number based on the spacing is so small as to make the convective acceleration negligible. In (87) and (88), y is measured in a direction normal to the walls, which are located at $y = 0$ and $y = b$, and μ is assumed to be independent of y. Since b is assumed to be very much smaller than the representative scale in the x-z plane, the second derivatives of u with respect to x and z are neglected in comparison with its second derivative with respect to y. The space between the walls is called a Hele-Shaw cell, when it is implied that fluid is to flow through it.

Since b is small, u and w can be assumed to have, locally, the distribution of a Poiseuille flow as far as the y-direction is concerned—that is,

$$(u, w) = 6\frac{y}{b}\left(1 - \frac{y}{b}\right)(U, W),$$

in which U and W are functions of x and z only. Thus (87) and (88) become

$$\frac{12\mu}{b^2} U = -\frac{\partial p}{\partial x}, \tag{89}$$

$$\frac{12\mu}{b^2} W = -\frac{\partial p}{\partial z} - g\rho, \tag{90}$$

which are identical with (72) if $b^2/12$ is identified with k and U and W with u and w respectively. (Remember that $u' = \mu u/\mu_0$, $w' = \mu w/\mu_0$.) Thus a Hele-Shaw cell is a porous medium of permeability $b^2/12$, and all the developments in Sections 2 and 4, and part of 3 apply to Hele-Shaw cells. This is important because laboratory experiments can be done with Hele-Shaw cells much more easily than with other porous media like sand, clay, or plastic balls.

6. INSTABILITY DUE TO A DIFFERENCE IN VISCOSITY

In the petroleum industry it is sometimes the practice to inject water into the oil field at certain spots in the attempt to drive oil to certain other spots for pumping. In this practice the phenomenon of fingering has long been recognized. The water, which is intended to push the oil forward from behind, tends to form fingers that penetrate through the oil to move forward, and from these big fingers little fingers soon develop, so that after a while the water assumes the form of a tree. This can be observed in a square Hele-Shaw cell originally filled with oil, with water pushing from behind, as shown in Figs. 58 and 59, at two stages of development.

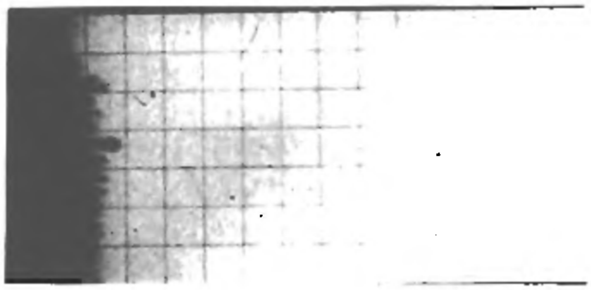


FIGURE 58. An early stage of penetration of a viscous fluid by a less viscous one, in a Hele-Shaw cell. (*Courtesy of Professor C. Weinaug.*)

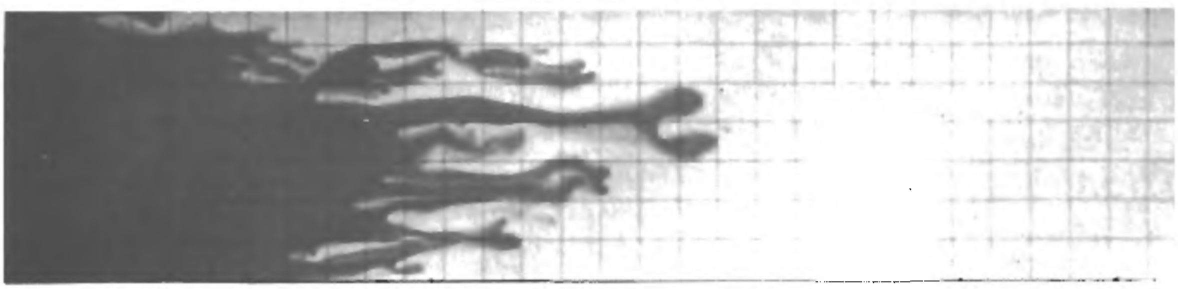


FIGURE 59. Advanced stage of penetration of a viscous fluid by a less viscous one, in a Hele-Shaw cell. Note the treelike formation. (*Courtesy of Professor C. Weinaug.*)

That water, being less viscous, tends to move ahead of the oil can perhaps be surmised from the results of Section 3, in which it is shown that fluid masses tend to follow a path of least resistance. Saffman and Taylor [1958] considered the stability of a semi-infinite fluid (Fluid 1) in contact with and being pushed upward by another semi-infinite fluid (Fluid 2), both moving with velocity U in the direction of the vertical, which is taken to be the x-direction. Since the velocity at the interface in the x-direction (which is normal to it) must be the same for both fluids, and since the perturbations to the mean flow must vanish at infinity for both fluids, the appropriate forms for the potentials are

$$\phi_1 = Ux - \frac{a\sigma}{n} \exp (iny - nx + \sigma t),$$

$$\phi_2 = Ux + \frac{a\sigma}{n} \exp (iny + nx + \sigma t),$$

in which n is the wave number for the y-direction, σ is the rate of amplification, t is the time, and a is the amplitude of the two-dimensional corrugations of the interface. These potentials satisfy the Laplace equation and the boundary conditions. Now

$$p_1 = -\frac{\mu_1}{k_1}\phi_1 - \rho_1 g x,$$

and

$$p_2 = -\frac{\mu_2}{k_2}\phi_2 - \rho_2 g x.$$

The continuity of pressure at the interface then demands that

$$\frac{\sigma}{n}\left(\frac{\mu_1}{k_1} + \frac{\mu_2}{k_2}\right) = (\rho_1 - \rho_2)g + \left(\frac{\mu_1}{k_1} - \frac{\mu_2}{k_2}\right)U, \tag{91}$$

in which k_1 and k_2 may differ for the two fluids, even though the medium is the same. For one thing, some of Fluid 1 may be left behind with the medium, thus changing its permeability to Fluid 2. Thus, the flow is stable or unstable according as

$$\left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right)U + (\rho_2 - \rho_1)g \geq 0 \quad \text{or} \quad < 0.$$

It is interesting that the stability condition involves not only the differences in ρ and in μ, but also U. Thus a lighter fluid on top tends to stabilize the flow, as expected. As far as the effect of viscosity is concerned, a less viscous fluid in front is conducive to stability.

Staffman and Taylor also considered the stability of the same flow in a Hele-Shaw cell, taking the interfacial tension T into account. This tension causes a pressure difference across the interface of the amount $T(2/b + 1/R)$, in which R is the radius of curvature of the interface in the plane of motion. The amount $2T/b$ is constant, and does not enter into the stability condition, and the remaining term T/R can be written as $T(d^2x/dy^2)$ in a linear theory. A calculation similar to that done in the last paragraph yields the result

$$\frac{12}{b^2}\sigma(\mu_1 + \mu_2) = \frac{2\pi}{l}\left\{\frac{12U}{b^2}(\mu_1 - \mu_2) + g(\rho_1 - \rho_2)\right\} - \frac{8\pi^3 T}{l^3},$$

with the wavelength $l = 2\pi/n$, from which it can be seen that the flow is unstable for all wavelengths greater than

$$l_{\text{crit}} = 2\pi b T^{1/2}[12U(\mu_1 - \mu_2) + b^2 g(\rho_1 - \rho_2)]^{1/2}.$$

The flow is always stable if the quantity inside the brace is negative. The amplification factor σ is a maximum for $l = \sqrt{3}l_{\text{crit}}$.

Saffman and Taylor also examined the postinstability situation of fingering, and their treatment, supported by Taylor's beautiful experiments, is a triumph in simplicity. Figure 60 shows a finger developed from an original hump at the interface. The flow is described in Fig. 61. The finger moves as a solid body with speed U. Far to the right the fluid moves with speed V. The ratio V/U is denoted by λ. The complex potentials for the finger is $w_1 = \phi_1 + i\psi_1$, and that for the penetrated fluid is $w_2 = \phi_2 + i\psi_2$. Since the finger progresses with the speed U, the kinematic boundary condition on the interface is

$$\psi_1 = \psi_2 = Uy. \tag{92}$$

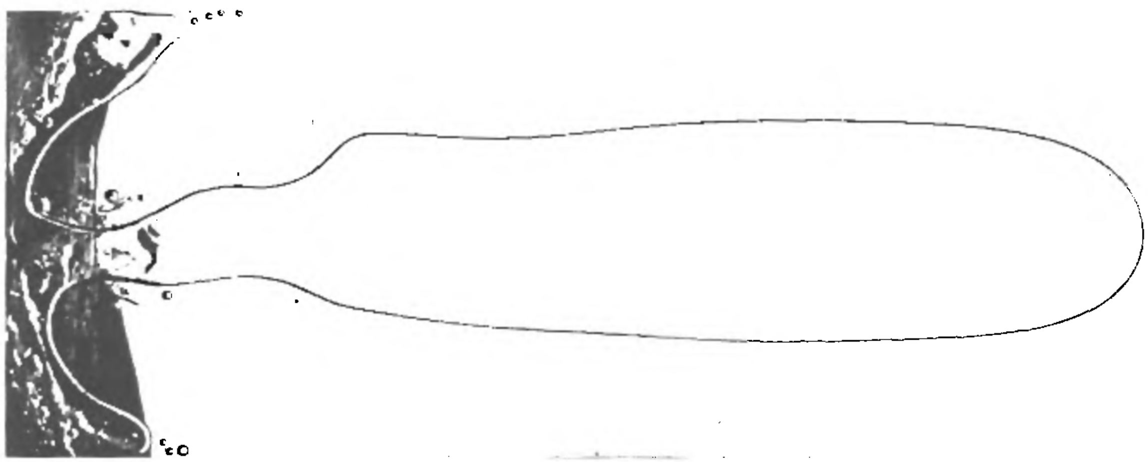


FIGURE 60. Development of a single air finger in a Hele-Shaw cell, after Saffman and Taylor [1958]. (*Proc. Roy. Soc.* (*A*), *245, pp. 312–329. Courtesy of the Royal Society of London.*)

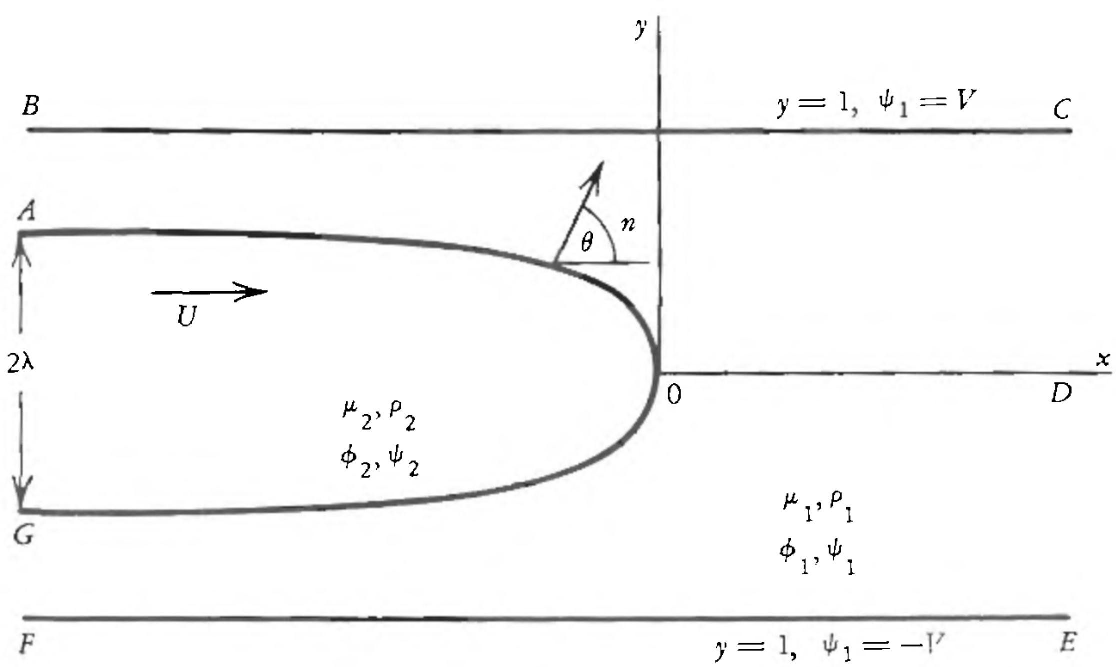


FIGURE 61. Schematic drawing for finger movement, after Saffman and Taylor [1958]. (*Proc. Roy. Soc.* (*A*), *245, pp. 312–329. Courtesy of the Royal Society of London.*)

If surface-tension effects are neglected, the condition for continuity of pressure is

$$(12/b^2)(\mu_1\phi_1 - \mu_2\phi_2) = g(\rho_2 - \rho_1)x. \tag{93}$$

The other boundary conditions are

$$\begin{aligned} \phi_1 &\to Vx \quad \text{as} \quad x \to +\infty, \\ \phi_2 &\to Ux \quad \text{as} \quad x \to -\infty, \\ \psi_1 &= \pm V \quad \text{on} \quad y = \pm 1. \end{aligned} \tag{94}$$

The boundaries of the Hele-Shaw cell are $y = \pm 1$. This amounts to using the half width of the cell as the scale of length, and expressing x and y in terms of it. Actually the condition $\phi_2 = Ux$ is true for the entire fluid constituting the finger.

If μ_2 is very small and gravity effects are neglected, and with ϕ_1 written as ϕ, (93) becomes

$$\phi = 0,$$

and the domain in the ϕ-ψ plane for the more viscous fluid is as shown in Fig. 62. With y as the dependent variable and ϕ and ψ as the independent ones, solution of the equation (by the method of separation of variables)

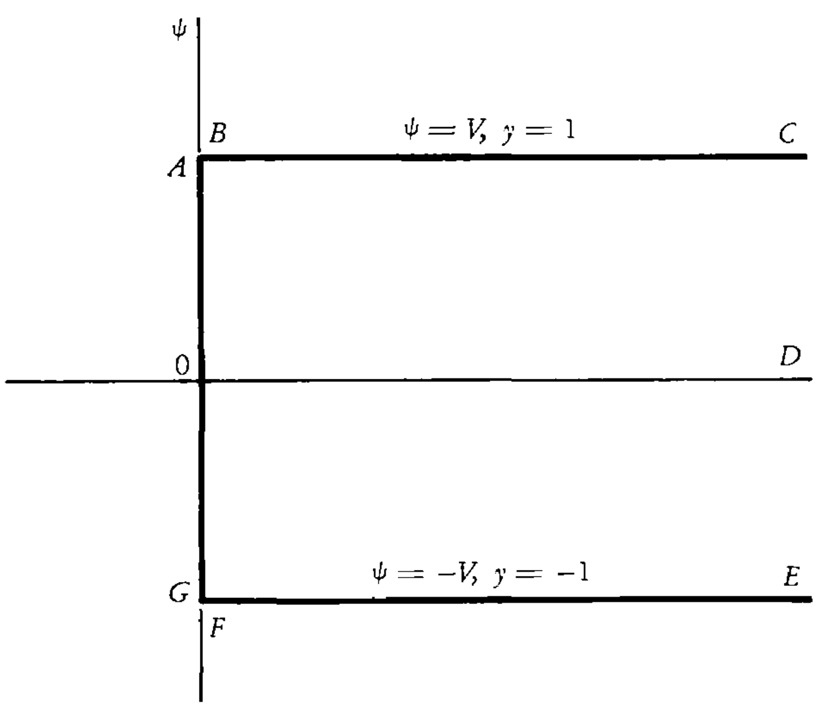


FIGURE 62. Complex-potential plane for finger movement, after Saffman and Taylor [1958]. (*Proc. Roy. Soc. (A), 245, pp. 312–329. Courtesy of the Royal Society of London.*)

with the boundary conditions (92) and $y = \pm 1$ at $\psi = \pm V$ yields the result

$$y = \frac{\psi}{V} + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi\psi}{V} \exp\left(-\frac{n\pi\phi}{V}\right), \tag{95}$$

with

$$A_n = (-1)^n \frac{2}{n\pi}\left(1 - \frac{V}{U}\right) = (-1)^n \frac{2}{n\pi}(1 - \lambda).$$

Equation (95) is the imaginary part of the equation

$$z = \frac{w}{V} + \frac{2}{\pi}(1 - \lambda) \ln \frac{1}{2}\left[1 + \exp\left(-\frac{\pi w}{V}\right)\right], \tag{96}$$

with $z = x + iy$ and $w = \phi + i\psi$. (The factor $\frac{1}{2}$ is to make $z = 0$ at $w = 0$.) On setting $\phi = 0$ and $\psi = Uy$ in (96), the equation of the interface is obtained:

$$x = \frac{1 - \lambda}{\pi} \ln \frac{1}{2}\left(1 + \cos \frac{\pi y}{\lambda}\right). \tag{97}$$

The solution just obtained can be generalized so that (93) in its unsimplified form is exactly satisfied. The complex potential for the fluid inside the finger is still

$$\phi_2 + i\psi_2 = U(x + iy).$$

If U^* is defined by

$$\frac{12\mu_1 U^*}{b^2} = g(\rho_1 - \rho_2) - \frac{12\mu_2 U}{b^2},$$

the boundary condition for ϕ_1 on the interface becomes $\phi_1 = -U^*x$, exactly. The new complex potential

$$W = \Phi + i\Psi = w_1 + U^*z$$

then transforms the boundary conditions to

$$\begin{aligned} &\Phi = 0, \quad \Psi = (U + U^*)y \quad \text{on the interface,} \\ &\Psi = \pm(V + U^*) \quad \text{on} \quad y = \pm 1, \\ &W \to (V + U^*)z \quad \text{as} \quad x \to +\infty. \end{aligned}$$

Thus the more general problem can be reduced to the simpler problem solved in the last paragraph by identifying $V + U^*$, $U + U^*$, and W with V, U, and w in that paragraph, respectively. The half width of the bubble is given by

$$\lambda = \frac{V + U^*}{U + U^*},$$

and the maximum velocity of the bubble is

$$U_{\max} = \frac{\mu_1}{\mu_2} V + \frac{b^2 g}{12\mu_2}(\rho_1 - \rho_2),$$

corresponding to $\lambda = 0$, as in the simpler case.

Since the parameter λ is arbitrary in the theory, the solution given by Saffman and Taylor is not unique. Experiments done by them have brought out several interesting points:

(1) In many experiments with various fluids and speeds U, λ was very nearly $\frac{1}{2}$, and the calculated profiles for $\lambda = \frac{1}{2}$ agreed beautifully with the photographed profiles.

(2) As U was decreased, λ increased, and the agreement in profile shape became less satisfactory. A photograph of a bubble shape for $\lambda = 0.87$ did not agree at all with the calculated profile. The reason for the disagreement is not yet known precisely, but surface tension or other surfaces stresses, and the amount of fluid left behind by the fluid in front of the bubble must have something to do with the shape of the bubble.

(3) If μ_2 was very much smaller than μ_1, as when air or water was used for the bubble, λ was uniquely determined by the parameter $\mu_1 U/T$. As the latter was varied from zero to 0.15, λ decreased from 1 to $\frac{1}{2}$, and remained $\frac{1}{2}$ for any larger values of $\mu_1 U/T$.

Many people have attempted to explain why the value $\frac{1}{2}$ for λ is apparently "favored by the bubble." No satisfactory explanation is yet available.

The stability criterion given by (91) is based on (1). If, instead of (1), we use

$$\frac{\rho}{\varepsilon}\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \rho X_i - \frac{\mu}{k} u_i,$$

the flow, if started from rest, is still irrotational. The equation determining σ is

$$\sigma^2 A + \sigma B = nQ,$$

in which

$$A = \frac{\mu_1}{k_1} + \frac{\mu_2}{k_2}, \qquad B = \frac{\rho_1}{\varepsilon_1} + \frac{\rho_2}{\varepsilon_2},$$

$$Q = (\rho_1 - \rho_2)g + \left(\frac{\mu_1}{k_1} - \frac{\mu_2}{k_2}\right)V.$$

Thus

$$\sigma = \frac{-B \pm \sqrt{B^2 + 4nAQ}}{2A},$$

from which it again follows that $Q = 0$ gives the condition for neutral stability. The formula for σ given by Taylor and Saffman is still correct if AQ is small. It is interesting that for the same n there are two values of σ, one of which corresponds to a highly damped mode. This mode is not of physical significance since it is not easily observable and since the stability is governed by the other mode.

The case of a bubble progressing horizontally in a vertical Hele-Shaw cell has been treated by Matar [1963].

Gravitational instability of a viscous fluid in a porous medium and heated from below has been considered by Lapwood [1948] and Wooding [1960a, b]. Wooding's first problem [1960a] differs from Lapwood's only in that a vertical boundary is present in the former, which affects the character of the mode of motion and hence the stability parameter.

7. MAINTAINED GRAVITATIONAL CONVECTION FROM A LINE OR POINT SOURCE

The significance of gravitational convection in ground-water flow has been pointed out by Wooding [1963]. Since solutions of nonlinear problems in seepage flow are very few, and the problem of gravitational convection from a source is a nonlinear one, it seems desirable to present a brief version of the solution here, especially since Wooding stated that his solution is valid for Prandtl number equal to 1 only.

Consider first the case of a two-dimensional heat or mass source in infinite space. In a plane normal to its length, let the trace of the source be the origin of Cartesian coordinates x and z, with z measured in the direction of the vertical. The equations of motion for steady flows can be written

$$\frac{\mu}{k} u = -\frac{\partial p}{\partial x}, \tag{98}$$

$$\frac{\mu}{k} w = -\frac{\partial p}{\partial z} - \rho g, \tag{99}$$

in which u and w are the gross-velocity components in the directions of increasing x and z, respectively, μ is the viscosity and ρ the density of the fluid, p is the pressure, g is the gravitational acceleration, and k is the permeability of the porous medium. The equation of diffusion for steady two-dimensional flow in a porous medium is

$$u \frac{\partial s}{\partial x} + w \frac{\partial s}{\partial z} = \kappa\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial z^2}\right), \tag{100}$$

in which s is the deviation of either the temperature or the concentration from a standard value, and κ is the diffusivity. Equations (98) and (99) are the generalized law of Darcy for steady flow, and Eq. (100) is based on the assumption of constant porosity. The more general equations applicable to unsteady flows and variable porosity are to be found in Wooding [1959, p. 121]. Furthermore, the diffusivity κ includes both molecular diffusivity and intersticial diffusion. Since the latter has directional properties dependent upon the velocity field, the assumption of the constancy of κ serves only to give an approximate solution to the problem. For a discussion of intersticial diffusion, see Saffman [1959 and 1960].

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{101}$$

since compressibility effects can be neglected. Hence Lagrange's stream function can be used, in terms of which

$$u = \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial \psi}{\partial x}. \tag{102}$$

Whether s denotes a temperature deviation or a concentration deviation, so long as it is not too large, the density can be expressed as

$$\rho = \rho_0[1 - \alpha(s - s_0)], \tag{103}$$

in which ρ_0 is the density of the fluid at a standard temperature or concentration s_0. The coefficient α is the expansivity if s denotes a temperature, and can be either positive or negative if s denotes a concentration. For definiteness, we shall henceforth consider the source to be a heat source. The applicability of the results or equations to mass diffusion is understood.

Elimination of p from (98) and (99) by cross differentiation and utilization of (102) and (103) produces

$$\frac{\mu}{k}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\psi = -g\alpha\rho_0 \frac{\partial s}{\partial x}. \tag{104}$$

The equation of diffusion can be written in the form

$$\frac{\partial \psi}{\partial z}\frac{\partial s}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial s}{\partial z} = \kappa\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial z^2}\right). \tag{105}$$

For the region not too near the source, longitudinal diffusion can be neglected in comparison with transverse diffusion, so that (105) can be written as

$$\frac{\partial \psi}{\partial z}\frac{\partial s}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial s}{\partial z} = \kappa \frac{\partial^2 s}{\partial x^2}, \tag{106}$$

which is of the boundary-layer type. Similarly, (104) can be written as

$$\frac{\mu}{k}\frac{\partial^2 \psi}{\partial x^2} = -g\alpha\rho_0 \frac{\partial s}{\partial x}. \tag{107}$$

The neglect of $\partial^2 s/\partial z^2$ and $\partial^2 \psi/\partial z^2$ can be justified *a posteriori*, as will be done.

Integrating (106) with respect to x between $-\infty$ and ∞, and remembering that $s \to 0$ and $u = \partial\psi/\partial z$ approaches a finite value (to be shown later) as $x \to \pm\infty$, we have

$$-\frac{\partial}{\partial z}\int_{-\infty}^{\infty} s \frac{\partial \psi}{\partial x} = 0, \tag{108}$$

or, after division by s_0 (which corresponds to ρ_0),

$$-\int_{-\infty}^{\infty} \frac{s}{s_0}\frac{\partial\psi}{\partial x}\,dx = \int_{-\infty}^{\infty} \frac{sv}{s_0}\,dx = \text{constant } G. \tag{109}$$

This constant G is really the convective part of the strength of the source (per unit length in a direction normal to the x-y plane). With the transformations

$$\frac{\psi}{G} = \left(\frac{gz^3}{\nu^2}\right)^{1/9} f(\eta), \tag{110}$$

$$\frac{s}{s_0} = \left(\frac{gz^3}{\nu^2}\right)^{-1/9} \theta(\eta), \tag{111}$$

and

$$\eta = \left(\frac{gz^3}{\nu^2}\right)^{1/9} \frac{x}{z}, \tag{112}$$

in which ν is the kinematic viscosity and equal to μ/ρ_0, (106) and (107) assume the forms

$$-(f\theta)' = \frac{3\kappa}{G}\theta \qquad \text{and} \qquad A\theta' + f'' = 0, \tag{113}$$

in which

$$A = \frac{s_0 \alpha k}{G}\left(\frac{g^2}{\nu}\right)^{1/3}. \tag{114}$$

The boundary conditions are

$$f(0) = 0, \qquad f''(0) = 0, \qquad f'(\pm\infty) = 0, \qquad \theta(\pm\infty) = 0. \tag{115}$$

The first two conditions are the consequences of symmetry, and the last two follow from the requirements that v and Δs must vanish at infinity, where $\rho = \rho_0$ and $s = s_0$. Note that $f''(0) = 0$ also implies $\theta'(0) = 0$.

A first integration of the two equations in (113) produces

$$-f\theta = \frac{3\kappa}{G}\theta' \qquad \text{and} \qquad A\theta + f' = 0, \tag{116}$$

the constants of integration being zero by virtue of the conditions at $\eta = 0$ and at infinity. From (116) it follows that

$$-ff' = \frac{3\kappa}{G} f'', \tag{117}$$

integration of which produces

$$\frac{6\kappa}{G} f' = B^2 - f^2, \tag{118}$$

B^2 being a constant of integration. Let

$$f = BF. \tag{119}$$

Then

$$1 - F^2 = \frac{6\kappa}{GB} F'. \tag{120}$$

This can be integrated directly to produce

$$\frac{GB}{6\kappa}\eta = \frac{1}{2}\ln\frac{1 + F}{1 - F} = \tanh^{-1} F \qquad (0 \le F < 1), \tag{121}$$

so that

$$F(\eta) = \tanh\frac{GB}{6\kappa}\eta. \tag{122}$$

The constant B is to be determined from (109), or from

$$-\int_{\infty}^{\infty} f'\theta\, d\eta = 1. \tag{123}$$

Since

$$f' = BF' = \frac{GB^2}{6\kappa}\operatorname{sech}^2\frac{GB}{6\kappa}\eta \tag{124}$$

and

$$\theta = -\frac{1}{A}f', \tag{125}$$

(123) becomes

$$\frac{1}{A}\left(\frac{GB^2}{6\kappa}\right)^2\int_{-\infty}^{\infty}\operatorname{sech}^4\frac{GB}{6\kappa}\eta\, d\eta$$

$$= \frac{G^2B^3}{6\kappa s_0\alpha k}\left(\frac{\nu}{g^2}\right)^{1/3}\int_{-\infty}^{\infty}\operatorname{sech}^4 z\, dz = 1,$$

or

$$\frac{2G^2B^3}{9\kappa s_0\alpha k}\left(\frac{\nu}{g^2}\right)^{1/3} = 1.$$

Hence

$$B = \left(\frac{9\kappa s_0\alpha k}{2G^2}\right)^{1/3}\left(\frac{g^2}{\nu}\right)^{1/9}. \tag{126}$$

The solution consists of (110), (119), (122), and (126), and θ is given by (125) and (114). It is the same as the Schlichting-Bickley solution for a two-dimensional jet, as Wooding [1963] noted for the special case of $\nu = \kappa$. It is actually valid for any Prandtl number, as has been shown. Wooding's experiments have verified the main features of his solution. Equations (102), (110), (119), (122), and (126) show that $u \neq 0$ at $x = \pm\infty$.

Since

$$\psi = G\left(\frac{gz^3}{\nu^2}\right)^{1/9} f\left[\left(\frac{gz^3}{\nu^2}\right)^{1/9}\frac{x}{z}\right], \tag{127}$$

it follows that, with $z_1 = (g/\nu^2)^{1/3}z$,

$$\frac{\partial^2\psi}{\partial x^2} \sim z_1^{-1} \quad \text{and} \quad \frac{\partial^2\psi}{\partial z^2} \sim z_1^{-5/3}.$$

Hence for large z

$$\frac{\partial^2\psi}{\partial z^2} \ll \frac{\partial^2\psi}{\partial x^2}. \tag{128}$$

Similarly,

$$\frac{\partial^2 s}{\partial x^2} \sim z_1^{-5/3} \quad \text{and} \quad \frac{\partial^2 s}{\partial z^2} \sim z_1^{-7/3},$$

so that for large z

$$\frac{\partial^2 s}{\partial z^2} \ll \frac{\partial^2 s}{\partial x^2}. \tag{129}$$

The assumptions made in (106) and (107) are therefore justified.

The flow pattern can best be obtained by plotting curves of constant values of ψ/κ in the X-Z plane, with

$$\frac{\psi}{\kappa} = 6Z^{1/3} \tanh(XZ^{-2/3}),$$
$$X = \left(\frac{GB}{6\kappa}\right)^3 \left(\frac{gx^3}{\nu^2}\right)^{1/3}, \quad Z = \left(\frac{GB}{6\kappa}\right)^3 \left(\frac{gz^3}{\nu^2}\right)^{1/3}. \tag{130}$$

The flow pattern is shown in Fig. 63. From (130) it can be seen that the greater GB/κ and g/ν^2 are, the more concentrated the plume will be in the actual flow. If one examines the variables in (126), one can conclude that the greater the values of $s_0\alpha$, k, G and g, and the smaller the values of ν and κ, the more concentrated the plume—an entirely reasonable situation.

The general pattern of isotherms can be obtained from (111), which now assumes the form

$$s_* = \frac{s}{s_0}\frac{A}{B}\left(\frac{6\kappa}{GB}\right)^2 = Z^{-1/3}\,\text{sech}^2(XZ^{-2/3}). \tag{131}$$

The pattern is shown in Fig. 64.

For axisymmetric convection a more exact similarity solution is possible. Unfortunately the resulting differential equations are not readily solvable. The formulation is presented here for another purpose—to point out the lack of consistency in the use of the boundary-layer equations in certain well known solutions.

With r and z denoting cylindrical coordinates, and z measured in the direction of the vertical, the equations of motion for axisymmetric flow are

$$\frac{\mu}{k}u = -\frac{\partial p}{\partial r}, \tag{132}$$

$$\frac{\mu}{k}w = -\frac{\partial p}{\partial z} - g\rho, \tag{133}$$

in which u and w denote gross-velocity components in the directions of increasing r and z. The equation of continuity is

$$\frac{\partial(ru)}{\partial r} + \frac{\partial(rw)}{\partial z} = 0, \tag{134}$$

which permits the use of Stokes' stream function ψ, in terms of which

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad w = \frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{135}$$

Elimination of p between (132) and (133) and utilization of (135) produces

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\psi}{\partial z^2} = \frac{g\alpha k}{\nu}\frac{\partial s}{\partial r}, \tag{136}$$

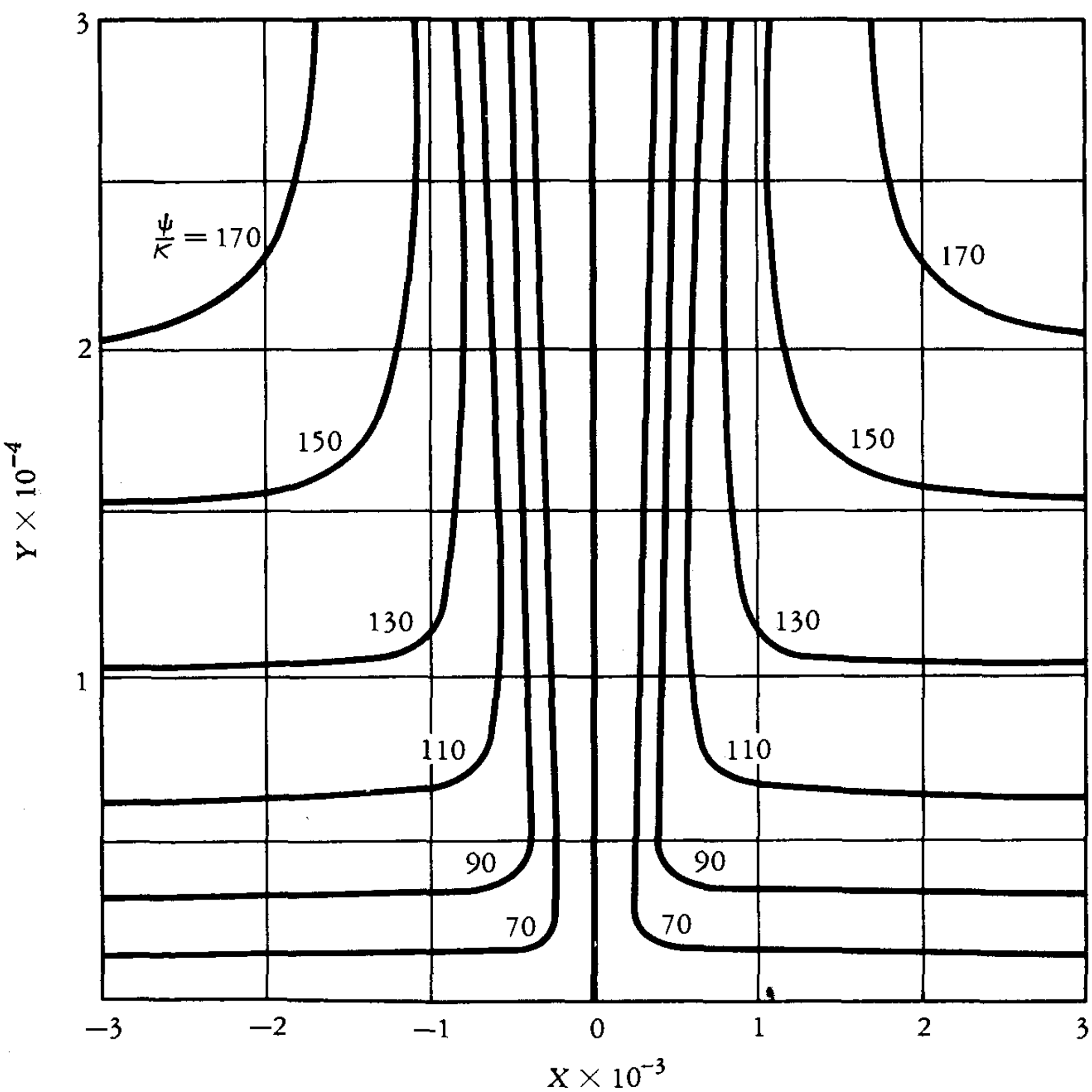


FIGURE 63. Pattern of two-dimensional convection in a porous medium from a boundary source.

in which s is related to ρ through (103). The diffusion equation is

$$u\frac{\partial s}{\partial r}+w\frac{\partial s}{\partial z}=\kappa\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)+\frac{\partial^2 s}{\partial z^2}\right], \tag{137}$$

or

$$-\frac{\partial\psi}{\partial z}\frac{\partial s}{\partial r}+\frac{\partial\psi}{\partial r}\frac{\partial s}{\partial z}=\kappa\left[\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)+r\frac{\partial^2 s}{\partial z^2}\right]. \tag{138}$$

Multiplication of (137) by $2\pi r$ or (138) by 2π and integration with respect to r produces, in view of the boundary conditions $u = 0$ at $r = 0$ and $s = 0$ at $r = \infty$,

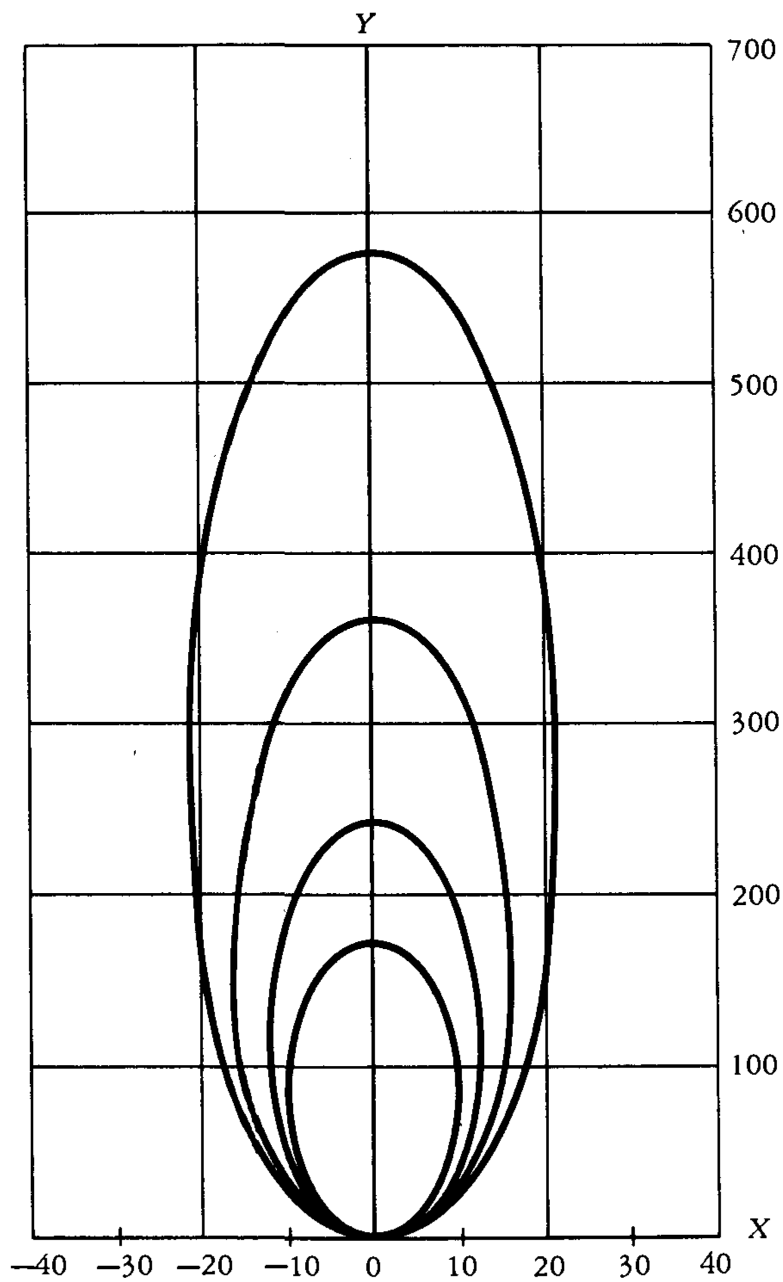


FIGURE 64. Isotherms of two-dimensional convection in a porous medium from a boundary source.

$$\frac{\partial}{\partial z}\left[\int_0^\infty 2\pi wsr\,dr - \kappa\int_0^\infty 2\pi r\frac{\partial s}{\partial z}\,dr\right] = 0. \tag{139}$$

The first integral indicates the heat flux by convection and the second, with the minus sign, that by conduction. From (139) it follows that

$$\frac{2\pi}{s_0}\int_0^\infty \left(ws - \kappa\frac{\partial s}{\partial z}\right) r\,dr = \text{constant } G. \tag{140}$$

The appropriate transformations are

$$\frac{\psi}{G} = \left(\frac{gz^3}{\nu^2}\right)^{1/3} f(\eta), \tag{141}$$

$$\frac{s}{s_0} = \left(\frac{gz^3}{\nu^2}\right)^{-1/3} \theta(\eta), \tag{142}$$

in which

$$\eta = \frac{r}{z}. \tag{143}$$

The form of (143) indicates that now the terms involving $\partial^2\psi/\partial z^2$ and $\partial^2 s/\partial z^2$ must be retained. In previous work of other authors [Schlichting, 1933; Görtler, 1954b] and of this writer [Yih, 1950] on axisymmetric jets with and without swirl, longitudinal diffusion of momentum, moment of momentum, or heat, has been neglected. Such neglect is not justifiable when the exponent of z in the expression for η is -1 as in (143), or less than -1. It is justified only when it is greater than -1. In fact, the terms neglected by all the three authors are of equal magnitude with those they retained. Schlichting's result for the round jet has been compared with Squire's exact result [1951] by students at the University of Michigan. They have found that the two results do not always agree, and the discrepancy can be quite large in certain ranges of the parameters, though it is small if the momentum flux is large.

The equation of motion then assumes the form

$$\frac{f''}{\eta} - \frac{f'}{\eta} + \eta f'' = \frac{\alpha s_0 k}{G}(\nu g)^{1/3}\theta'. \tag{144}$$

The diffusion equation is of the form

$$-(f\theta)' = \frac{\kappa}{G}\left(\frac{\nu^2}{g}\right)^{1/3}[(\theta'\eta)' + (\eta^2\theta)' + (\eta^3\theta')']. \tag{145}$$

The boundary conditions are

$$f(0) = \theta'(0) = 0,$$

$$\eta f'' - f' = 0(\eta^3) \quad \text{near} \quad \eta = 0, \quad \text{corresponding to} \quad \frac{\partial w}{\partial r} = 0$$

$$\text{at} \quad \eta = 0, \quad f'(\infty) = 0, \quad \text{and} \quad \theta(\infty) = 0.$$

Equations (144) and (145) can be integrated once to produce

$$\frac{f'}{\eta} + \eta f' - f = \frac{\alpha s_0 k}{G}(vg)^{1/3}\theta + C, \tag{146}$$

and

$$-f\theta = \frac{\kappa}{G}\left(\frac{v^2}{g}\right)^{1/3}(\theta'\eta + \eta^2\theta + \eta^3\theta'). \tag{147}$$

The constant of integration in (147) is zero because $f = 0$ at $\eta = 0$. Further integration of (146) and (147) appears difficult. The results have been presented here chiefly to make a point that appears to have been overlooked by similarity-solution seekers.

NOTES

Section 4.4

1. Koh [1964] gave a solution to the problem of axisymmetric flow of a viscous, diffusive, and linearly stratified fluid into a point sink. When he attempted to solve the corresponding problem for the line sink, he was troubled that the deviation s of the density from the linear profile does not vanish at infinite y, positive or negative, y being the vertical coordinate. List [1968] noted that this deviation, though finite, is very small. He deemed it negligible and proceeded to give Koh's solution, which he pronounced correct, in "a somewhat more formal fashion." What is puzzling is that Koh was perplexed by and List perpended the finite deviation of the density from the linear profile at infinite y, but neither of them seemed to be troubled by the negative density and negative salinity at sufficiently large positive y, becoming negative infinite at positive infinite y, as demanded by the linear profile assumed.

The linear density profile, so often assumed for the sake of convenience by so many people, is not a weakness peculiar to the cited works of Koh and List. But Koh's and List's similarity solution of the boundary-layer type, which gives parabolic streamlines extending from the sink to infinity, depends crucially on the maintenance of a density gradient in the vertical direction. To realize this gradient it is necessary to impose constant densities at two horizontal boundaries. These planes of constant density would intersect the parabolic streamlines of the Koh-List solution and thus vitiate it, except possibly near the sink.

Since according to their solution the fluid enters the sink from all elevations and depths, any selective withdrawal is not in the usual sense. The originally light fluid, upon descending, acquires more salinity, and the originally heavy fluid, upon ascending, becomes less saline, so that what enters the sink has undergone much change in salinity on the way to it. The most obvious and most important conclusion of the Koh-List solution, not explicitly pointed out by them, is that the mean density of the fluid entering the sink is the density at the level of the sink. This conclusion can be reached without their solution since from the governing equations one can see that the horizontal velocity component u is even in y and the deviation s of the density from the *linear* profile is odd in y, and since longitudinal salt conduction at any section gives zero salt flux and the salt conduction at any two symmetrically placed parabolic streamlines above and below the sink gives a sum of zero. The conclusion remains valid when two symmetrically placed horizontal boundaries are present to impose the density gradient needed, thus rendering the problem more realistic.

2. List [1969] showed that the solution given in Section 4.4 remains valid if the two plane boundaries are kept at constant temperatures (upper boundary at the higher temperature) and if diffusivity is taken into account.

Chapter 6

ANALOGY BETWEEN GRAVITATION AND ACCELERATION

I. INTRODUCTION

The analogy between the flow of a fluid with density variation and fluid flow in an accelerating or rotating frame of reference has been generally recognized. In the eighteenth century, d'Alembert already recognized that the mass-times-acceleration term in Newton's famous equation could be moved to the other side of the equation and be considered as a force, and in fact a body force of $-a$ per unit mass, acting in the direction opposite to that of the acceleration, which has the magnitude a. The body is then in equilibrium and at rest in a frame of reference accelerating with the body. In this noninertial system the acceleration; which may vary with time and space, manifests itself in the form of body forces. In a linearly accelerating system, the acceleration manifests itself in the form of gravity. In a rotating system, the centripetal acceleration and the Coriolis acceleration take the form of the centrifugal force and the Coriolis force, respectively, both of which are body forces. Since density and entropy stratifications, continuous or discontinuous, give rise to striking flow phenomena in the presence of gravity, we can expect these stratifications and the stratification of other quantities (such as circulation) to give rise to similar phenomena in an accelerating or rotating system. This chapter is devoted to the analogy between the flow of a fluid of variable density or entropy in a gravitational field and the flow of a fluid with variable density, entropy, or circulation in an accelerating or rotating system. This analogy helps the understanding of seemingly widely different phenomena by giving them a sort of unity, and stimulates the mind to discover new results in one field at the suggestion of similar results in the other.

2. EQUATIONS OF MOTION FOR AN ACCELERATING SYSTEM

The equations of motion of an inviscid fluid with respect to a fixed coordinate system are, in the notation of Chapter 1,

$$\rho\left(\frac{\partial}{\partial t} + u_\alpha \frac{\partial}{\partial x_\alpha}\right)u_i = -\frac{\partial p}{\partial x_i} + \rho X_i. \tag{1}$$

Viscosity could be included in the discussion. Since such an inclusion is not really relevant to the main point to be made here, it will be left out. The viscous terms can be added if desired.

If one wishes to describe the motion of the fluid relative to its container which is moving with constant acceleration, for instance, it is desirable to use a frame of reference which is fixed with respect to the container, and hence is itself accelerating. Let the acceleration of the frame of reference be $a_i(t)$ ($i = 1, 2, 3$), and the coordinates in that system be x_i'. Then

$$x_i' = x_i - \int_0^t \int_{C_i}^{t_1} a_i(t_2)\, dt_2\, dt_1, \tag{2}$$

provided at $t = 0$ the coordinates x_i' and the fixed coordinates x_i momentarily coincide. For clarity the time associated with x_i' will be denoted by t'. We have $t = t'$, but

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \int_{C_i}^t a_i(t)\, dt \cdot \frac{\partial}{\partial x_i'}. \tag{3}$$

The velocity components u_i' is related to u_i by

$$u_i = u_i' + \int_{C_i}^t a_i\, dt, \tag{4}$$

in which C_i determines the container velocity at $t = 0$. Substitution of (2), (3), and (4) into (1) yields, after cancellation of two terms,

$$\rho\left(a_i + \frac{\partial u_i'}{\partial t} + u_\alpha' \frac{\partial u_i'}{\partial x_\alpha'}\right) = -\frac{\partial p}{\partial x_i'} + \rho X_i.$$

After the accents on u_i' and x_i' are dropped, this equation can be written as

$$\rho\left(\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}\right) = -\frac{\partial p}{\partial x_i} + \rho(X_i - a_i). \tag{5}$$

Thus, in the moving frame of reference, the equations of motion are unchanged except that the three components of the body force per unit mass is reduced by a_1, a_2, and a_3, in accordance with d'Alembert's idea.

3. EQUATIONS OF MOTION FOR A ROTATING SYSTEM

The vector form of Eqs. (1) can be written as

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] = -\nabla p + \rho \mathbf{X}, \tag{6}$$

which, in cylindrical coordinates (r, θ, z), takes the following form:

$$\rho\left(\frac{Du}{Dt} - \frac{v^2}{r}\right) = -\frac{\partial p}{\partial r} + \rho X_r, \tag{7}$$

$$\rho\left(\frac{Dv}{Dt} + \frac{uv}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho X_\theta, \tag{8}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho X_z, \tag{9}$$

in which u, v, and w are used to denote the velocity components in the directions of increasing r, θ, and z, respectively, the X's are the body-force components, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}. \tag{10}$$

With respect to a frame of reference rotating with the angular speed $\omega(t)$ about the z-axis, the velocity components are

$$u' = u, \qquad v' = v - \omega r, \qquad w' = w. \tag{11}$$

The cylindrical coordinates r', θ', and z', fixed with respect to the rotating frame of reference, are related to r, θ, and z by

$$r' = r, \qquad \theta' = \theta - \int_0^t \omega\, dt, \qquad z' = z. \tag{12}$$

The variable t is the same for both sets of coordinates. Substitution of (11) and (12) into (7), (8), and (9) produces, after cancellation of some terms,

$$\rho\left(\frac{Du}{Dt} - \frac{v^2}{r} - 2\omega v - \omega^2 r\right) = -\frac{\partial p}{\partial r} + \rho X_r, \tag{13}$$

$$\rho\left(\frac{Dv}{Dt} + \frac{uv}{r} + 2\omega u\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho X_\theta, \tag{14}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho X_z, \tag{15}$$

after the accents on t', r', θ', z', u', v', and w' have been dropped wherever they occur. Remember they do in

$$\frac{D'}{D't} = \frac{\partial}{\partial t'} + u'\frac{\partial}{\partial r'} + \frac{v'}{r'}\frac{\partial}{\partial \theta'} + w'\frac{\partial}{\partial z'}.$$

Equations (13), (14), and (15) can be written in the vector form [Morgan, 1951]

$$\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p + \rho\mathbf{X} - 2\rho\boldsymbol{\omega}\times\mathbf{v} - \rho\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}). \tag{16}$$

in which $\mathbf{v}$ is the velocity vector and $\mathbf{r}$ the coordinate vector. In Cartesian coordinates, the equations are

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho X + 2\rho\omega v + \rho\omega^2 x, \tag{17}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho Y - 2\rho\omega u + \rho\omega^2 y, \tag{18}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho Z, \tag{19}$$

in which X, Y, and Z are written for X_x, X_y, and X_z, and u, v, and w are now Cartesian components of the velocity. Note that, unless there is axial symmetry, a steady flow in a rotating frame of reference is unsteady in a fixed frame, and vice versa.

The components $2\rho\omega v$ and $-2\rho\omega u$, in (13) and (14) as well as in (17) and (18), are the Coriolis force components. The quantity $\omega^2 r$ is the magnitude of the centrifugal force, and the components $\omega^2 x$ and $\omega^2 y$ are its x and y components, all per unit mass.

The term $\rho\mathbf{X}$ in (16) can be absorbed in the term $-\nabla p$, if the body force has a potential Ω. Furthermore, the last term in (16) can be absorbed in it. If r is the radial distance from the axis of rotation, (16) can be written as

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla\left(p + \rho\Omega - \frac{\rho\omega^2 r^2}{2}\right) - 2\rho\boldsymbol{\omega}\times\mathbf{v}. \tag{16a}$$

If the acceleration term on the left is so small that no appreciable error is committed in setting it equal to zero, the motion is governed by the equation

$$\nabla\left(p + \rho\Omega - \frac{\rho\omega^2 r^2}{2}\right) = -2\,\rho\boldsymbol{\omega}\times\mathbf{v}. \tag{16b}$$

Large-scale motion on the earth is often governed by (16b), and if so it is called geostrophic. In (16a) and (16b), ρ is assumed constant.

In geophysical applications the term $\rho\omega^2 r^2/2$ is small compared with $\rho\Omega$ (Ω is g times the vertical distance) after the gradient has been taken. It can therefore be neglected. Written out in local Cartesian coordinates at a latitude ϕ on the earth rotating with angular velocity ω, (16b) has the form

$$\frac{\partial p}{\partial x} = 2\rho\omega(\sin\phi)v, \tag{17a}$$

$$\frac{\partial p}{\partial y} = -2\rho\omega(\sin\phi)u, \tag{18a}$$

$$\frac{\partial p}{\partial z} = -\rho g. \tag{19a}$$

Thus, geostrophic motions are governed by the hydrostatic equation in the direction of the vertical, and, in a horizontal plane, is characterized by the fact that the pressure gradient produces the Coriolis acceleration exactly—or balances the Coriolis force exactly.

Equations (17a) and (18a) have far-reaching consequences. They state that in any horizontal plane the velocity is perpendicular to the gradient of pressure, which is to say that it is parallel to the isobars—a remarkable property! In fact, the horizontal velocity "leads" the pressure gradient (in a horizontal plane) by 90°. If the streamlines are closed in a horizontal plane in the northern hemisphere, a clockwise circulation always encloses a high-pressure region, and a counterclockwise circulation encloses a low-pressure region, looking from above. The reverse is true in the southern hemisphere.

4. STIFFENING EFFECT OF VORTEX LINES

We shall now discuss briefly the theorem of Proudman [1916], which states that steady weak flows in a rotating fluid are two-dimensional. The word "steady" in this theorem implies that the flow is steady relative to a body moving with *constant velocity with respect to the coordinate axes rotating with the fluid.*

Consider a body moving in a fluid rotating with constant angular speed ω, with a constant velocity u_0 relative to the fluid, in the x-direction. The axes of the coordinates x, y, and z rotate with the fluid. Let the coordinates x', y', and z' be for a frame of reference fixed with respect to the body. Then

$$x = x' + u_0 t', \qquad y = y', \qquad z = z', \qquad t = t',$$

if x and x' have the same value at $t = 0$, and

$$u = u' + u_0, \qquad v = v', \qquad w = w',$$

in which u', v', and w' are velocity components with respect to the coordinates x', y', and z'. The term Du/Dt in (17) is

$$\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\right)u' + u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y} + w' \frac{\partial u'}{\partial z}.$$

Now

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x},$$

and by assumption of steadiness with respect to the body $\partial u'/\partial t' = 0$. If u', v', and w' are so small that quadratic terms in them can be neglected (which implies that u_0 should be small), $\rho(Du/Dt)$ in (17) can be neglected. Similarly, the left-hand sides of (18) and (19) can be neglected. The equations of motion in the coordinates x, y, and z then become

$$\frac{\partial P}{\partial x} = 2\omega v, \qquad \frac{\partial P}{\partial y} = -2\omega u, \qquad \frac{\partial P}{\partial z} = 0, \tag{20}$$

in which

$$P = \frac{p}{\rho} - \frac{1}{2}\omega(x^2 + y^2) + \Omega,$$

with Ω denoting the body-force potential. Differentiation of the first two equations in (20) with respect to z produces

$$\frac{\partial v}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial z} = 0,$$

in virtue of the third equation in (20). Furthermore, cross differentiation of the first two equations in (20) produces

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Since the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

it follows that

$$\frac{\partial w}{\partial z} = 0.$$

Thus u, v, and w are all independent of z. If a weak motion is generated steadily in a rotating fluid at a certain level, there is a tendency for the fluid cylinder enclosing the body causing the motion, with axis parallel to the z-axis, to have exactly the same motion at all levels. This result was due to Proudman [1916], and verified experimentally by Taylor [1923].

For steady axisymmetric motion which differs from solid body rotation only by a weak perturbation, the governing equations are

$$\frac{\partial p}{\partial r} = -2\omega v, \qquad u = 0, \qquad \frac{\partial p}{\partial z} = 0,$$

in which u, v, and w are now velocity components in the directions of increasing cylindrical coordinates r, θ, and z. The equation $u = 0$ follows from axisymmetry, since

$$2\omega u = -\frac{1}{r}\frac{\partial p}{\partial \theta} = 0,$$

there being no body force in the direction of increasing θ. The equation of continuity is

$$\frac{\partial(ru)}{\partial r} + \frac{\partial(rw)}{\partial z} = 0. \tag{21}$$

Since $u = 0$,

$$\frac{\partial w}{\partial z} = 0.$$

Thus the radial component of the velocity is completely inhibited, and the axial velocity is independent of z. This train effect was discovered by Proudman [1916]. Experiments done by Taylor [1923] and Long [1953] verified Proudman's prediction fairly well for the fluid in front of the moving body, but not so well for the fluid in the rear. The disagreement is probably due to the finiteness of the disturbance created and to viscosity.

The stiffening effect of vortex lines in a rotating fluid is directly analogous to the stiffening effect of isopycnic lines in a stratified fluid. See Section 3, Chapter 1. These stiffening effects are further analogous to the stiffening effect of magnetic lines of force [see Yih, 1959e].

5. CENTRIFUGAL WAVES

The equations governing small unsteady axisymmetric motion relative to solid-body rotation are, in cylindrical coordinates,

$$\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial r} + 2\omega v, \qquad \frac{\partial v}{\partial t} = -2\omega u, \qquad \frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z}. \tag{22}$$

The equation of continuity permits the use of Stokes' stream function ψ, in terms of which

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad w = \frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{23}$$

Elimination of P and v from (22) produces, after (23) is used,

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\psi + \left(4\omega^2 + \frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 \psi}{\partial z^2} = 0. \tag{24}$$

If the small unsteady motion relative to the rotating frame of reference is oscillatory in nature, the time dependence of ψ can be assumed to be of the form $e^{i\sigma t}$. Then (24) takes the form

$$\sigma^2\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\psi + (\sigma^2 - 4\omega^2)\frac{\partial^2\psi}{\partial z^2} = 0, \tag{25}$$

or

$$4\sigma^2\eta\frac{\partial^2\psi}{\partial\eta^2} + (\sigma^2 - 4\omega^2)\frac{\partial^2\psi}{\partial z^2} = 0, \quad \eta = r^2. \tag{26}$$

Equation (26) is hyperbolic if $\sigma^2 < 4\omega^2$, and elliptic if $\sigma^2 > 4\omega^2$ [Görtler, 1944]. In the former case real characteristic surfaces exist. In the latter case ψ can be determined in an r-z region only if its value or its normal derivative, or a combination of the two, is prescribed on a closed curve in the meridianal plane enclosing that region. In this regard the flow resembles irrotational flows, which are also governed by a differential equation of the elliptic type. If the oscillations have a low frequency, σ is small, and the characteristics are almost parallel to the z-axis. Görtler pointed out that this fact is equivalent to Proudman's result in Section 4.

Comparing this situation with that for waves in a stratified fluid, as discussed at the end of Section 15 of Chapter 2, we see that there is very close analogy between waves in a fluid with general solid-body rotation, which may be called centrifugal waves for brevity, and gravity waves in a stratified fluid.

If the dependence of ψ on z is sinusoidal, and

$$\psi = f(\eta)\, e^{ikz}, \tag{27}$$

in which k is the wave number, (26) assumes the form

$$f'' - \frac{k^2}{4\eta}\left(1 - \frac{4\omega^2}{\sigma^2}\right)f = 0. \tag{28}$$

If the radius of the cylinder containing the fluid is a, the boundary conditions are

$$f(0) = 0 \qquad \text{and} \qquad f(a^2) = 0. \tag{29}$$

Equations (28) and (29) form a Sturm-Liouville system. Given the wave number k, there are infinitely many eigenvalues for σ^{-2}, corresponding to as many modes of wave motion. As $\sigma^{-2} \to \infty$, the number of zeros increases indefinitely. Thus, for the same k, the speed of propagation σ/k decreases to zero as the number of zeros in the interval $0 \leq r \leq a$ increases indefinitely. In the case of a wave maker, σ is fixed, and there are infinitely many eigenvalues for k, some of which are real and the rest imaginary. It can also be seen from (28) that for a given k and radius a, σ increases directly with ω. That is to say the "stiffer" the vortex lines of the undisturbed fluid are, the faster the speed of wave propagation along them. One can, in a rough way, liken the quantity ω^2 with the tension in a string. If $\sigma > 2\omega$, all k's are imaginary.

The existence, reality, and distribution of the eigenvalues σ^{-2} (for fixed k), or of the eigenvalues k^2 (for fixed σ), the partition of kinetic and "potential" energies, the separability of the kinetic and "potential" energies of the normal modes, and the estimate and approximate calculation of the eigenvalues can be discussed in exactly the same manner as in Chapter 2. Again, the two situations are analogous.

It can be readily shown that axisymmetric hydromagnetic waves in an infinitely conducting fluid with axial current are governed by a Sturm-Liouville system like (28) and (29), provided that the walls are such as to allow linear homogeneous boundary conditions. These waves may be called centripetal waves, because the electromagnetic body force produced by the magnetic field and the current is centripetal. They are analogous to centrifugal waves and to gravity waves in a stratified fluid.

5.1. *Rossby Waves*

Although Rossby waves are not closely analogous to gravity waves, they will be discussed here briefly because of their geophysical significance. Equations (17) and (18), with the centripetal acceleration terms neglected, and written in local Cartesian coordinates on the earth, are

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + 2\rho\omega(\sin\phi)v, \tag{17b}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} - 2\rho\omega(\sin\phi)u, \tag{18b}$$

Since w is neglected, (17b) and (18b) can be written as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(u, v) = \frac{1}{\rho}\left(-\frac{\partial p}{\partial x}, -\frac{\partial p}{\partial y}\right) + (fv, -fu),$$

in which $f = 2\omega \sin\phi$. Since w is assumed zero, if u and v are supposed independent of y, u is also independent of x, because incompressibility is assumed. Thus u can only be a function of t. If it is also assumed independent of t, and p is eliminated, the following equation results:

$$\frac{\partial^2 v}{\partial t\,\partial x} + u\frac{\partial^2 v}{\partial x^2} + \beta v = 0,$$

in which $\beta = \partial f/\partial y$ is a constant at any given value of z (vertical distance from the surface of the earth). This equation in v possesses the solution

$$v = A\,e^{i\alpha(x-ct)},$$

in which

$$c = u - \frac{\beta}{\alpha^2},$$

α being the wave number. The waves represented by this solution are called Rossby waves.

If vertical motion is not neglected, gravity waves can occur, and they may be mixed with Rossby waves. The governing equations are still (17b) and (18b), except that now $\partial u/\partial x + \partial v/\partial y$ is no longer zero. The linear development is straightforward. For linear and nonlinear filtering of Rossby waves from long gravity waves, see Thompson [1961, pp. 76–79 and 79–81].

6. HYDRAULIC JUMP IN A ROTATING FLUID

One instance of the suggestiveness of the analogy between gravitation and acceleration is the discovery of the hydraulic jump in a rotating fluid [Yih, Gascoigne, and Debler, 1963]. The ordinary hydraulic jump, which is so well discussed in every book on elementary fluid mechanics that its discussion has

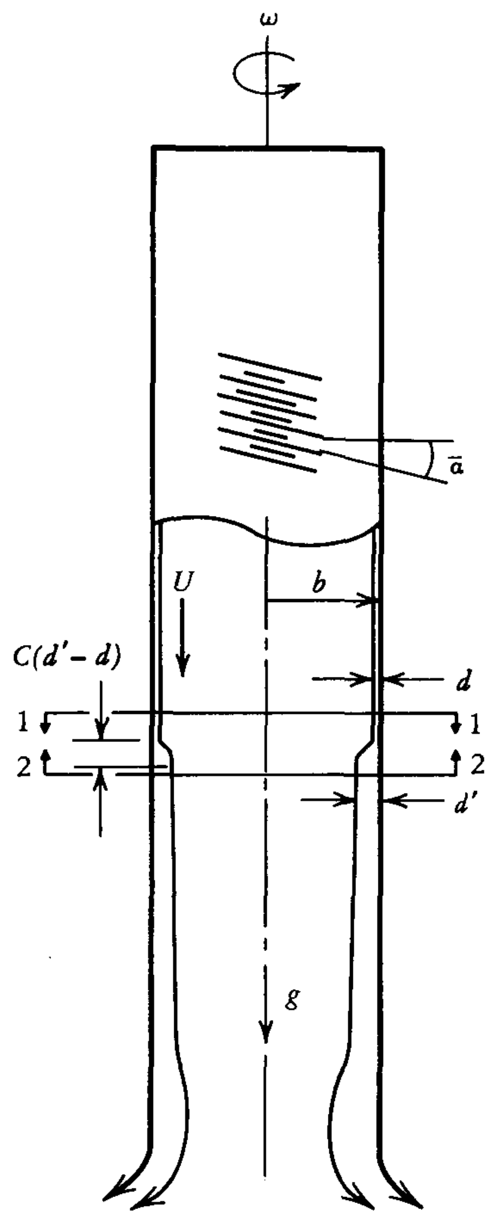


FIGURE 65. Definition sketch for hydraulic jump in a rotating liquid. (*Courtesy of The Physics of Fluids.*)

been omitted in this one, is a gravitational phenomenon. The analogy suggested that a similar phenomenon might well exist in a rotating fluid, provided there be a discontinuity in the density, as in the case of the hydraulic jump, at the free surface. (Actually, the discontinuity can be in the quantity $\rho\Gamma^2$, ρ being the density, Γ being the circulation at any radial distance r from the center line of the rotating cylinder. See Section 9.) The hydraulic jump being considered here occurs in a liquid film flowing vertically down the inner wall of a rotating cylinder, when the downstream control forces the film to thicken (Fig. 65). The jump occurs at the location where the depth just downstream of it as determined from the "back-water" curve traced upstream from the downstream section controlling the opening of the exit is exactly the same as demanded by the momentum and continuity considerations at the jump. A photograph of the jump is shown in Fig. 66, in which the position of the jump is clearly visible.

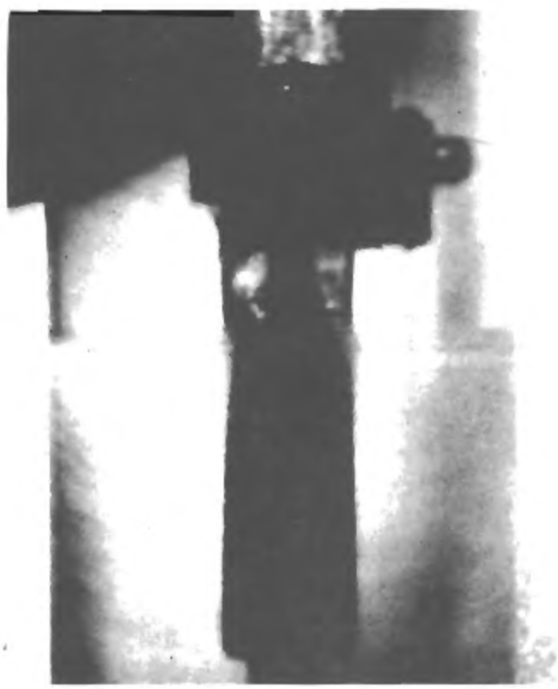


FIGURE 66. Photograph of a hydraulic jump in a rotating liquid. (*Courtesy of The Physics of Fluids.*)

The angular speed of the rotating cylinder is denoted by ω. The fluid upstream from the jump is assumed to have the same angular speed. The inner radius of the cylinder is denoted by b. The radius of the upstream free surface is denoted by a, and that downstream by a'. The upstream and downstream depths of the liquid at the jump are

$$d = b - a \qquad \text{and} \qquad d' = b - a'.$$

The pressure in the fluid upstream from the jump is

$$p = \frac{\rho\omega^2}{2}(r^2 - a^2). \tag{30}$$

Downstream from the jump, the angular speed ω' will in general vary from one radial position to another. If viscous and turbulent mixings are neglected, ω' can be determined from Kelvin's theorem on the circulation of a material ring, provided the upstream and downstream distributions of the axial velocity are known. Because in the gravitational hydraulic jump intense turbulent mixing is observed in the section of varying depth, the same can be expected to happen in the rotating hydraulic jump. Thus it is unrealistic to assume that Kelvin's theorem holds in the present case. The other extreme is the assumption of perfect mixing and the complete lack of any historical identity. This is not exactly valid, but is not likely to be far wrong. Thus ω' will be assumed constant. The pressure in the fluid downstream from the jump is thus

$$p' = \frac{\rho\omega'^2}{2}(r^2 - a'^2). \tag{31}$$

With reference to Fig. 65, the total axial force P acting at cross section 1—1 and that (P') acting at cross section 2—2 can be found from p and p' by integration:

$$P = \frac{\rho\pi}{4}\omega^2(b^2 - a^2)^2, \qquad P' = \frac{\rho\pi}{4}\omega'^2(b^2 - a'^2)^2. \tag{32}$$

The discharge is given by

$$Q = \int_a^b U\, 2\pi r\, dr = \int_{a'}^b U'\, 2\pi r\, dr,$$

in which U and U' are the axial velocity upstream and downstream, respectively. The velocities U and U' are not uniform, but for turbulent flow they are nearly so. On the assumption of their uniformity,

$$U(b^2 - a^2) = U'(b^2 - a'^2). \tag{33}$$

The momentum fluxes through cross sections 1—1 and 2—2 are, respectively,

$$M = \int_a^b \rho U^2\, 2\pi r\, dr, \qquad \text{and} \qquad M' = \int_{a'}^b \rho U'^2\, 2\pi r\, dr.$$

If U and U' are assumed constant,

$$M = \rho\pi U^2(b^2 - a^2) \qquad \text{and} \qquad M' = \rho\pi U'^2(b^2 - a'^2). \tag{34}$$

The angular momentum fluxes before and after the jump can be assumed equal, because the torque exerted by the wall of the cylinder can be neglected. Thus

$$\int_a^b (\rho\omega r^2)U\, 2\pi r\, dr = \int_{a'}^b (\rho\omega' r^2)U'\, 2\pi r\, dr, \tag{35}$$

in which ω is constant. If ω', U, and U' are assumed constant, (35) becomes

$$\omega(b^2 + a^2) = \omega'(b^2 + a'^2), \tag{36}$$

after (33) is used.

The momentum equation applied to the fluid between cross sections 1—1 and 2—2 is

$$P - P' + W = M' - M, \tag{37}$$

in which W is the weight of the fluid in the region of varying depth. If the depth is assumed to vary linearly from d to d' in that region, and if the length of the jump is assumed to be $c(d' - d)$ or $c(a - a')$, c being a constant of proportionality,

$$W = g\rho\pi c(a - a')[b^2 - aa' - \tfrac{1}{3}(a - a')^2],$$

after some simplifications. The momentum equation now becomes

$$\frac{\rho\pi}{4}[\omega^2(b^2 - a^2)^2 - \omega'^2(b^2 - a'^2)] + g\rho\pi c(a - a')[b^2 - aa' - \tfrac{1}{3}(a - a')^2]$$
$$= \rho\pi[U'^2(b^2 - a'^2) - U^2(b^2 - a^2)].$$

In virtue of (33) and (36), this can be written

$$(1 - \alpha^2\alpha'^2)(1 - \alpha^2) = F_1^2(1 - \alpha^2)(1 + \alpha'^2)$$
$$+ \frac{cG(1 - \alpha'^2)(1 + \alpha'^2)^2[3(1 - \alpha\alpha') - (\alpha - \alpha')^2]}{3(\alpha + \alpha')}, \tag{38}$$

in which

$$\alpha = \frac{a}{b}, \qquad \alpha' = \frac{a'}{b}, \qquad F_1 = \frac{U}{\omega b}, \qquad G = \frac{g}{b\omega^2}.$$

Equation (38) enables one to find α_2 in terms of α_1, F_1, G, and c.

If d and d' are small compared with b, α and α' are nearly equal to 1. Putting them equal to 1 in (38) except where differences are involved, we obtain

$$(1 - \alpha\alpha')(1 - \alpha') = 2F_1^2(1 - \alpha) + cG(1 - \alpha')[(1 - \alpha\alpha') - \tfrac{1}{3}(\alpha - \alpha')^2]. \tag{39}$$

With

$$\eta = \frac{d'}{d} \qquad \text{and} \qquad F^2 = \frac{U^2}{\omega^2 bd},$$

(39) becomes

$$\eta(\eta + 1)(1 - cG) + \frac{1}{3}\frac{d_1}{b}cG\eta(\eta - 1)^2 = 2F^2, \tag{40}$$

or, since d_1/b is assumed small,

$$\eta(\eta + 1)(1 - cG) = 2F^2, \tag{41}$$

provided $d\eta/b$ is small compared with 1. Of course the order of (41) is one less than that of (40). In using (41), one of the three roots of (40) is lost. But (41) can be used to check experimental data, provided they do correspond to a value of $d\eta/b$ which is small compared to 1. The solution of (41) is

$$\eta = \frac{d'}{d} = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{8F^2}{1 - cG}}\right]. \tag{42}$$

This formula is given by Yih, Gascoigne, and Debler [1963]. Their experimental data are presented in Fig. 67, which shows that (42) is valid in the main with $c = 7$, approximately. The full details of their experiment are presented in their paper. Hydraulic jumps in a swirling fluid were also observed by Binnie [1962].

$\frac{b\omega^2}{g} = \frac{1}{G}$	20-25	26-30	31-35	36-40	41-45	46-50
Symbol	○	□	△	x	▫	◇

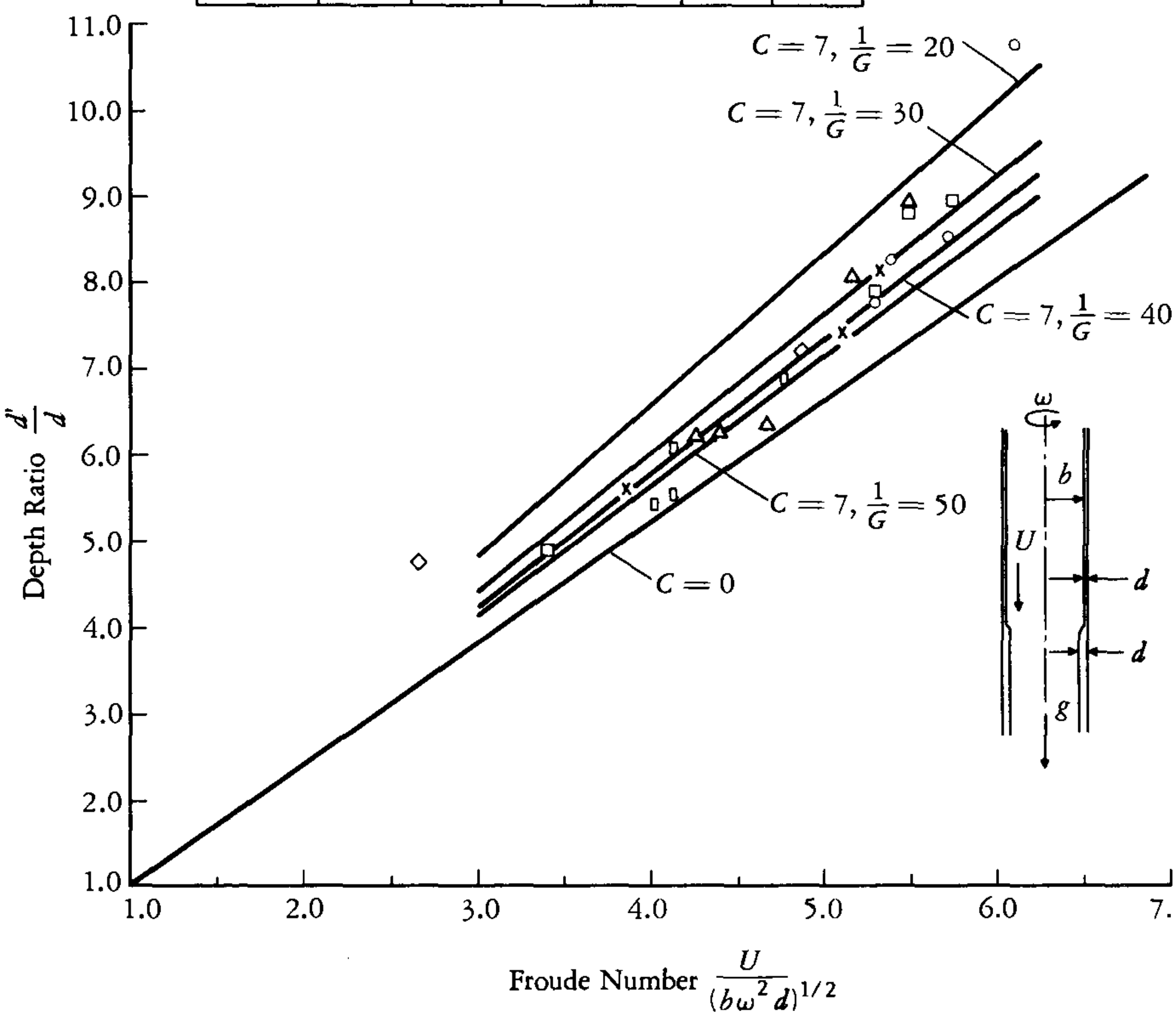


FIGURE 67. Comparison between theory and experiment for hydraulic jump in a rotating liquid. (*Courtesy of The Physics of Fluids.*)

Note that except for the term cG, which is necessary because the rotating cylinder was kept vertical to obtain axial symmetry, (42) is exactly the same as the corresponding formula for gravitational hydraulic jump. Although (42) has been derived for the stationary jump, it can be used to compute the speed of propagation of a moving bore in a rotating fluid, by the simple device of using a frame of reference moving with the jump, and thus making it stationary. It can be shown readily that a bore in a rotating fluid propagates faster if its amplitude $d' - d$ is greater—a situation entirely analogous to that for gravitational bores.

7. LARGE-AMPLITUDE STEADY FLOWS OF A SWIRLING FLUID

Like the steady flow of a stratified inviscid fluid, the steady flow of a swirling inviscid fluid is governed by an equation simple enough to be solvable. Since the equation is not obtained by linearization, but is exact, the amplitude of the motion does not have to be small.

We shall consider axisymmetric motion only, and fixed cylindrical coordinates r, θ, and z are used. With u, v, and w denoting the velocity components in the directions of increasing r, θ, and z, and with Ω denoting the body-force potential, the equations governing axisymmetric motion are

$$\frac{Du}{Dt} - \frac{v^2}{r} = -\frac{\partial}{\partial r}\left(\frac{p}{\rho} + \Omega\right), \tag{43}$$

$$\frac{Dv}{Dt} + \frac{uv}{r} = 0, \tag{44}$$

$$\frac{Dw}{Dt} = -\frac{\partial}{\partial z}\left(\frac{p}{\rho} + \Omega\right), \tag{45}$$

in which

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}. \tag{46}$$

Equation (44) can be written as

$$\frac{D}{Dt}(vr) = 0, \tag{47}$$

which expresses the conservation of angular momentum of a particle, or the conservation of circulation of a material ring, depending on the point of view. It is valid in general, whether the flow is steady or not. For steady flows a particle is always on the same streamline or rather the same stream surface. Hence (47) can be written

$$(vr)^2 = f(\psi),$$

in which ψ is the Stokes stream function. For steady flows, (43) and (45) can then be written as

$$w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) - \frac{f(\psi)}{r^3} = -\frac{\partial}{\partial r}\left(\frac{p}{\rho} + \frac{u^2 + w^2}{2} + \Omega\right),$$

$$u\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right) = -\frac{\partial}{\partial z}\left(\frac{p}{\rho} + \frac{u^2 + w^2}{2} + \Omega\right).$$

Cross differentiation of these equations and the use of (21) produces

$$\frac{D}{Dt}\frac{\zeta}{r} + \frac{1}{r^4}\frac{\partial f(\psi)}{\partial z} = 0, \tag{48}$$

in which ζ is the θ-component of the vorticity defined by

$$\zeta = \frac{\partial w}{\partial r} - \frac{\partial u}{\partial z} = \frac{1}{r}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi. \tag{49}$$

Equation (48) can be written as

$$\frac{D}{Dt}\frac{\zeta}{r} - u\frac{f'(\psi)}{r^3} = \frac{D}{Dt}\left(\frac{\zeta}{r} + \frac{f'(\psi)}{2r^2}\right) = 0,$$

with $f'(\psi) = df/d\psi$, or [Bragg and Hawthorne, 1950]

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi + \frac{f'(\psi)}{2} = r^2\, h(\psi), \tag{50}$$

which governs axisymmetric motion of a swirling inviscid fluid.

If at infinity ($z = \infty$) the longitudinal velocity w is constant and equal to $-W$ and the fluid is rotating with constant angular speed ω,

$$v = \omega r \qquad \text{and} \qquad \psi = -\frac{Wr^2}{2} \tag{51}$$

at infinity,

$$f(\psi) = \omega^2 r^4 = \frac{4\omega^2}{W^2}\psi^2 \qquad \text{and} \qquad h(\psi) = -\frac{2\omega^2}{W},$$

and (50) becomes

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi + \frac{4\omega^2}{W^2}\psi = -\frac{2\omega^2}{W}r^2, \tag{52}$$

which is the equation of Long [1953a]. Long [1953a, p. 200] gave the general solution of (52), which however violates (51). Long [1956c] realized this, for later, in dealing with the flow of a rotating fluid into a sink, he did not attempt to use his general solution for small values of the Rossby number. Such a use would violate (51). Long's solution, by the method of separation of variables, of the problem of the flow of a fluid in a rotating cylinder of radius b into a point sink is, with the origin of the coordinates at the sink,

$$\psi = -\frac{Wr^2}{2} + r\sum_{n=1}^{\infty} A_n \exp\left[-(\lambda_n^2 - Ro^{-2})^{1/2}\left(\frac{z}{b}\right)\right]\cdot J_1\left(\lambda_n\frac{r}{b}\right), \tag{53}$$

in which

$$Ro = \frac{W}{2\omega b} \tag{53a}$$

is the Rossby number, λ_n is the nth zero of $J_1(\lambda)$, and

$$A_n = -\frac{Wb}{\lambda_n J_0^2(\lambda_n)}. \tag{53b}$$

Solution (53) can be compared with Yih's solution for a stratified fluid—Eq. (40) of Chapter 3. If *Ro* is greater than $1/\lambda_1 = 0.261$, Long's solution does not violate the assumed upstream conditions. If *Ro* is less than $1/\lambda_0$, his solution ceases to be valid. A formal solution can still be given [Yih, O'Dell, and Debler, 1962] by relaxing the upstream condition. But it corresponds to the appearance of upstream waves, which have never been observed. Hence it will not be presented.

In reality, as Long's experiments [1953a, 1956c] show, for small Rossby numbers there is a stagnation region surrounding a core of flowing fluid. An analysis similar to that applied by Kao (see Section 8 of Chapter 3 on blocking) to a similar problem in stratified flow can be performed to determine (1) the critical Rossby number at which a stagnation zone will appear and (2) the flow pattern after the Rossby number is below critical.

For low Rossby numbers, stagnation zones can be presented by the instalment of special structures near the sink. The form of the axisymmetric structure is determined mathematically by changing the character of the singularity at the sink. Thus, for $1/\lambda_2 < Ro < 1/\lambda_1$, we can impose the following conditions at $z = 0$:

$$\begin{aligned} \psi &= -\frac{Wb^2}{2} \qquad \text{for} \quad \varepsilon \le r \le b, \\ \psi &= -\frac{Wb^2}{2} - \frac{Wb}{\lambda_1^4 J_0(\lambda_1)} k^3 r\, J_1\left(\frac{kr}{b}\right) \qquad \text{for} \quad 0 < r \le \varepsilon, \\ \psi &= 0 \qquad \text{for} \quad r = 0, \end{aligned} \tag{54}$$

in which

$$k\varepsilon/b = \lambda_1 = 3.8317,$$

and ε will be made to approach zero in the limit. Since

$$\lim_{\varepsilon \to 0} \int_0^{\varepsilon} k^3 r\, J_1\left(\frac{kr}{b}\right) J_1\left(\frac{\lambda_n r}{b}\right) dr = -\frac{\lambda_1^2 \lambda_n J_0(\lambda_1)}{2}$$

and

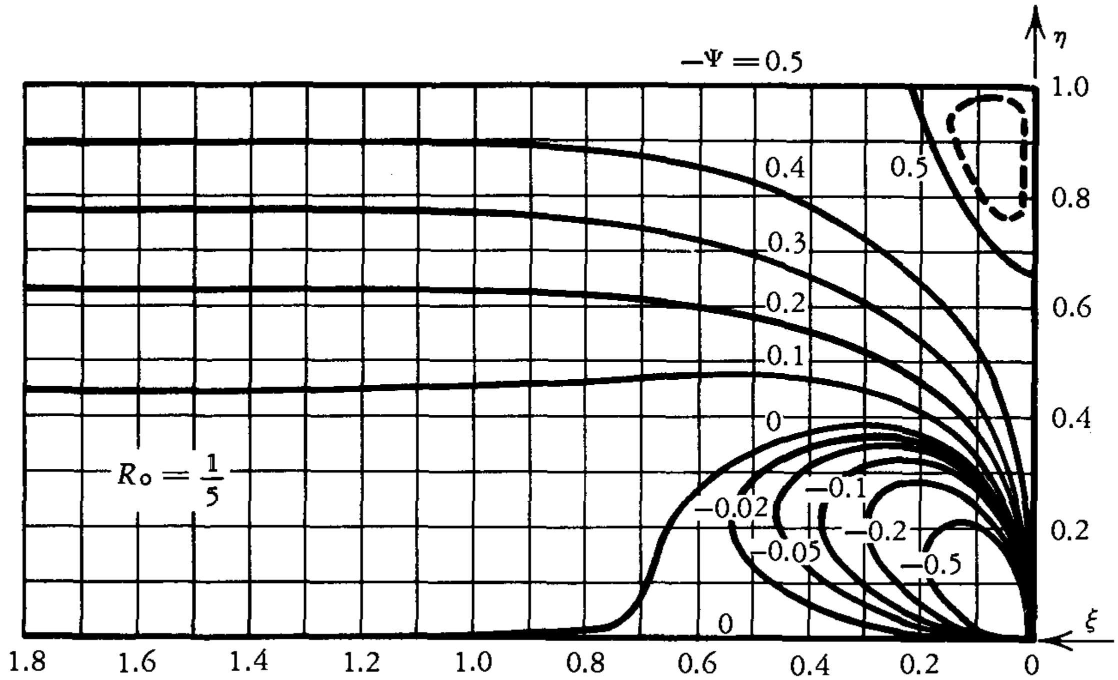
$$\int_0^b r\, J_1^2\left(\lambda_n \frac{r}{b}\right) dr = \frac{J_0^2(\lambda_n)}{2},$$

the Fourier-Bessel expansion of the term involving k in (54) is, in the limit,

$$-\frac{Wb}{\lambda_1^4 J_0(\lambda_1)} k^3 r\, J_1\left(\frac{kr}{b}\right) = \frac{Wb}{\lambda_1^2} \sum_{n=1}^{\infty} \frac{\lambda_n}{J_0^2(\lambda_n)} J_1\left(\frac{\lambda_n r}{b}\right) = \sum_{n=1}^{\infty} -\frac{\lambda_n^2}{\lambda_1^2} A_n J_1\left(\frac{\lambda_n r}{b}\right). \tag{55}$$

Hence the solution is

$$\psi = -\frac{Wr^2}{2} + r \sum_{n=2}^{\infty} A_n\left(1 - \frac{\lambda_n^2}{\lambda_1^2}\right) \exp\left[-(\lambda_n^2 - Ro^{-1})^{1/2}\left(\frac{z}{b}\right)\right] \cdot J_1\left(\lambda_n \frac{r}{b}\right). \tag{56}$$



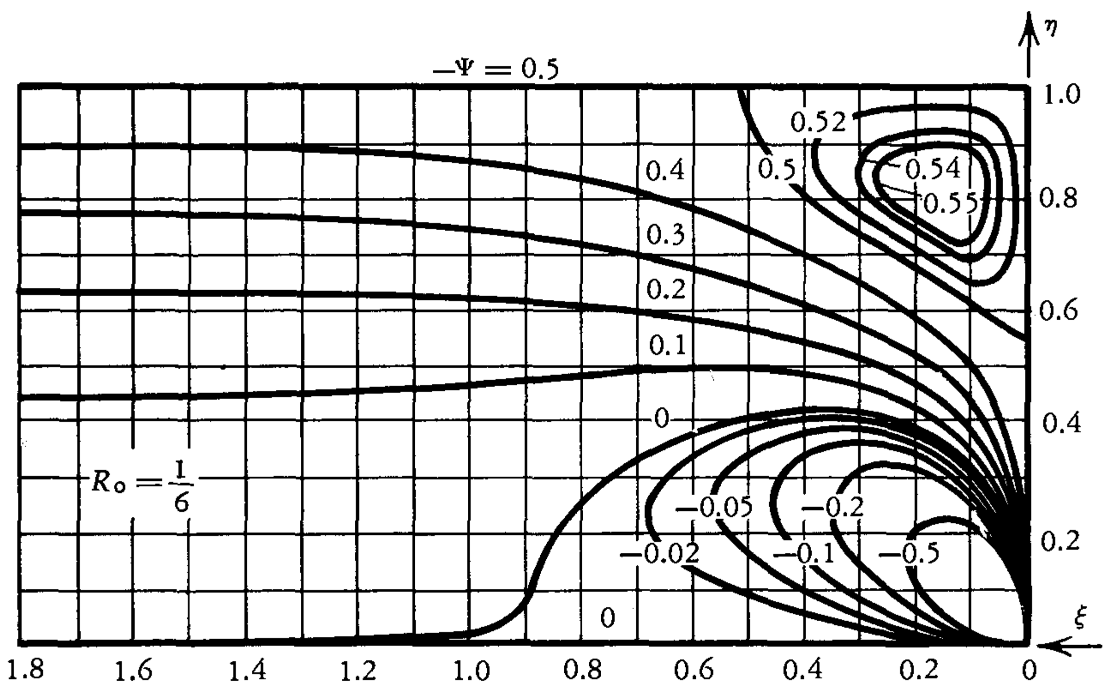
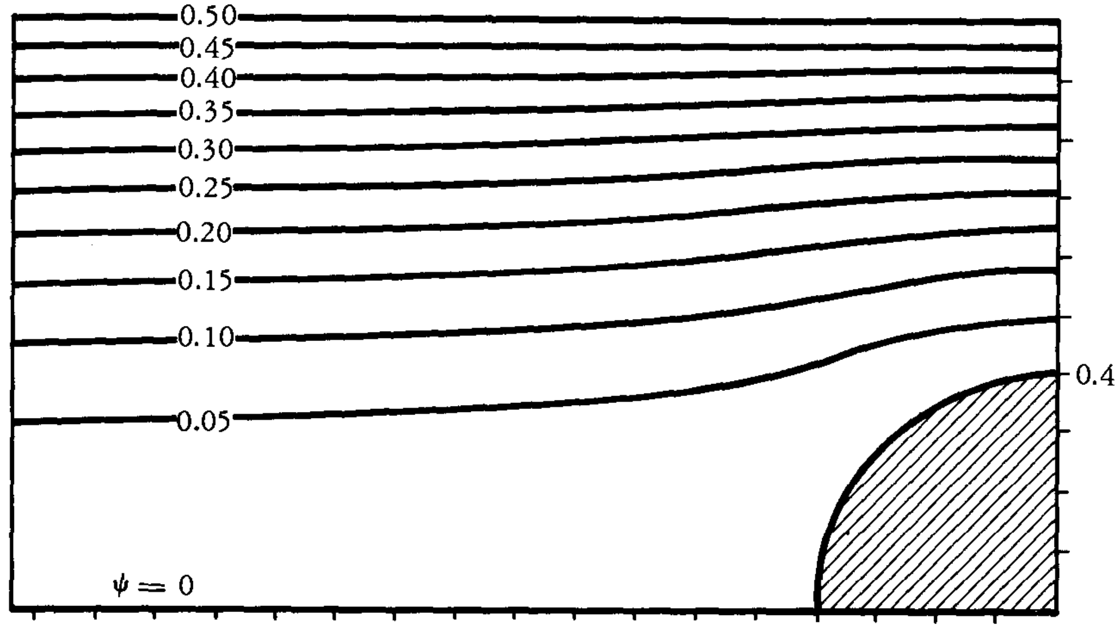


FIGURE 68. Flow of a rotating fluid into a sink at Rossby number $\frac{1}{5}$, with spindle structure near the sink marked by the streamline $\psi = 0.1$, $\xi = z/b$, $\eta = r/b$. (*Courtesy of the American Society of Mechanical Engineers.*)

FIGURE 69. Flow of a rotating fluid into a sink at Rossby number $\frac{1}{6}$, with spindle structure near the sink marked by the streamline $\psi = 0.2$. $\xi = z/b$, $\eta = r/b$. (*Courtesy of the American Society of Mechanical Engineers.*)

This solution was given by Yih [see Yih, O'Dell, and Debler, 1962]. It is similar to his solution for a stratified fluid, given in the same paper and presented in (3.50). For still lower Rossby numbers, solutions similar to (3.51) for stratified fluid can be constructed. For $Ro = \frac{1}{5}$ and $\frac{1}{6}$, both of which are between $1/\lambda_1$ and $1/\lambda_2$, flow patterns are given in Figs. 68 and 69, respectively. These patterns were obtained by O'Dell.



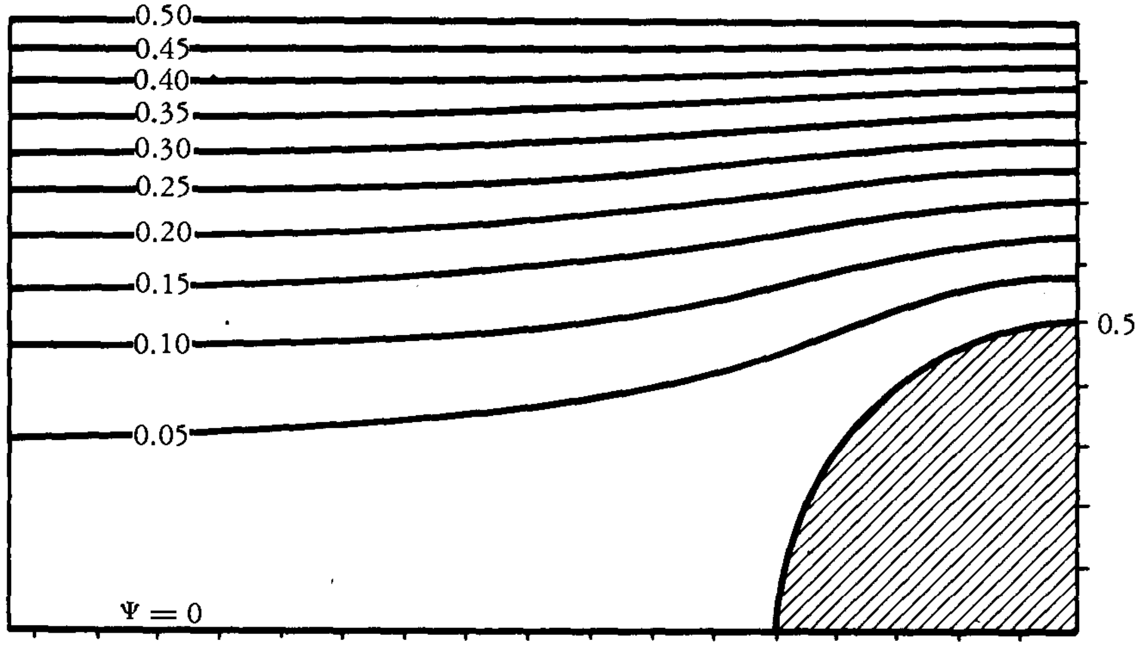


FIGURE 70. Pattern of swirling flow past a sphere inside a pipe, after Lai [1961 and 1964]. Rossby number = $\frac{1}{3}$. Radius of sphere = 0.4 times the radius of the pipe. (*J. Fluid Mech., 18, part 4. Courtesy of the Cambridge Univ. Press.*)

FIGURE 71. Pattern of swirling flow past a sphere inside a pipe, after Lai [1961 and 1964]. Rossby number = $\frac{1}{3}$. Radius of sphere = half that of the pipe. (*J. Fluid Mech., 18, part 4. Courtesy of the Cambridge Univ. Press.*)

Swirling flow in a pipe of radius b past a symmetrically located sphere has been studied by Wei Lai [1961 and 1964] with the method of Fraenkel [1956], which is similar in spirit to the method presented in Section 7 of Chapter 3, for stratified flow past a barrier. Three flow patterns obtained by Lai are shown in Figs. 70, 71, and 72. The Rossby number is defined by (53a), and the upstream conditions are as specified by (51).

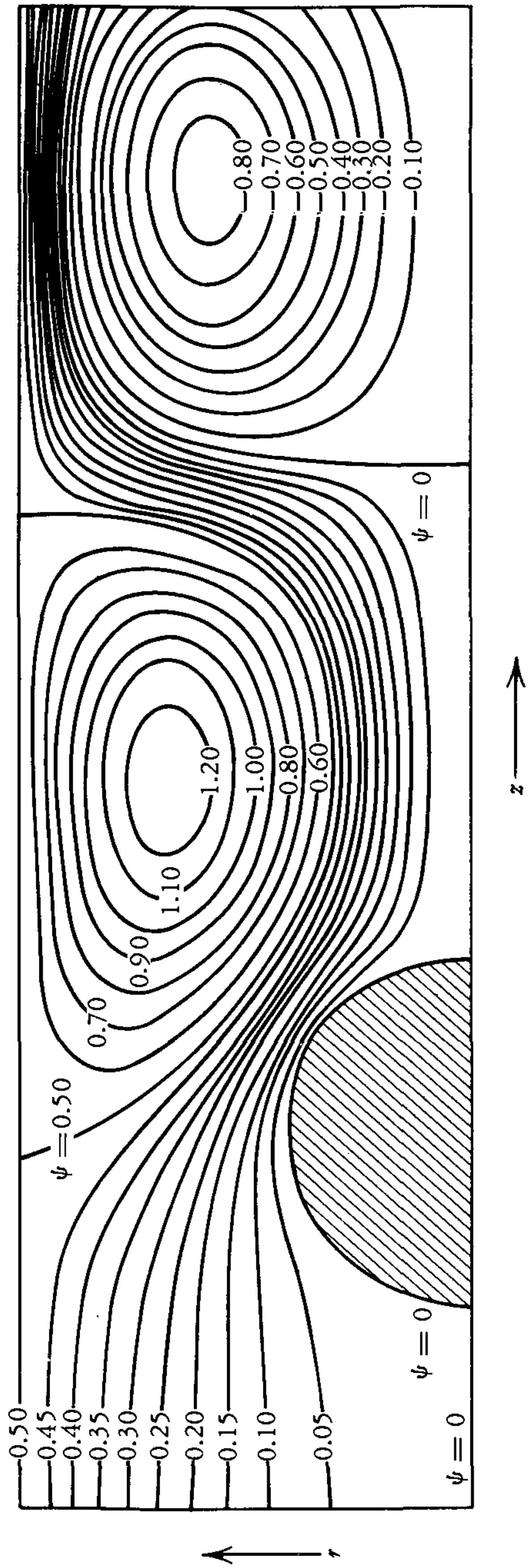


FIGURE 72. Pattern of swirling flow with lee waves past a sphere inside a pipe, after Lai [1961 and 1964]. Rossby number = $\frac{2}{9}$. Radius of sphere = 0.4 times radius of pipe. (*J. Fluid Mech.*, *18*, *part 4. Courtesy of the Cambridge Univ. Press.*)

8. STABILITY OF ACCELERATING FLUIDS

As has been seen in Section 1, if the container of a fluid accelerates with the acceleration a, the equations of motion relative to the container differ from those with respect to fixed coordinates only in that the body force per unit mass is modified by the amount $-a$. It is then understandable that an open vessel containing liquid is unstable if it accelerates downward with a constant acceleration a greater than g, for the situation is exactly the same as if it were turned upside down and the gravitational acceleration were $a - g$. If $a < g$, the fluid is stable. All this is obvious. But the situation is not so obvious if a is not constant but a function of time. Benjamin and Ursell [1954] showed that an open oscillating vessel containing water can be unstable even if the maximum downward acceleration never exceeds g. We shall see that the variability of acceleration makes the mechanism of instability wholly different from that in the case of constant acceleration. Thus the case of constant downward acceleration greater than g, which is strictly analogous to gravity instability, is relatively straightforward, whereas the case of varying acceleration, which would correspond to the unrealistic case of varying gravity, is full of subtleties. A physical interpretation of free-surface instability due to varying acceleration will be given after the problem has been formulated.

The equations of motion of a fluid, assumed inviscid, relative to a container moving with a varying acceleration a are, in Cartesian coordinates with z measured vertically upward from the bottom of the container,

$$\frac{D}{Dt}(u, v, w) = -\frac{1}{\rho}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)p + (0, 0, -g - a), \tag{57}$$

in which a is counted positive when the acceleration is upward, u, v, and w are the velocity components in the directions of increasing x, y, and z, respectively, and D/Dt stands for

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

Only the case of an acceleration varying with time will be considered. From (57), it can be seen that when the frame of reference is taken to be the container, the potential energy with respect to the container is based on $g + a$ instead of g. Although the concept of potential energy is meaningful only if a is constant, the explanation based on constant a, to be given later, is qualitatively applicable to the case of variable acceleration. The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{58}$$

Since the fluid is of constant density and assumed inviscid, and the apparent body force $g + a$ is conservative, irrotationality will persist. The motion under consideration can be considered to have started from rest, so that the subsequent motion is irrotational. For irrotational motion, the velocity can be expressed as the gradient of a potential ϕ:

$$(u, v, w) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi, \tag{59}$$

so that (58) becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0, \tag{60}$$

which is linear. The problem is then to solve (10) with the appropriate boundary conditions. But one of the boundary conditions, the one at the free surface, involves the pressure. It is therefore necessary to obtain an expression for p in terms of ϕ. This is supplied by the Bernoulli equation for irrotational unsteady flows:

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \frac{u^2 + v^2 + w^2}{2} + (g + a)z = F(t). \tag{61}$$

The function $F(t)$ is independent of x, y, and z, and can be absorbed in ϕ by adding to ϕ the function $\int F(t)\, dt$, without affecting either the velocity components or p. Hence, for convenience $F(t)$ will be taken to be zero.

The conditions at the rigid boundaries are

$$\frac{\partial\phi}{\partial z} = 0 \qquad \text{at the bottom,} \tag{62}$$

and

$$\frac{\partial\phi}{\partial n} = 0 \qquad \text{at the walls.} \tag{63}$$

If ζ denotes the displacement of the free surface from its undisturbed position, the free surface is

$$z = \zeta(x, y, t), \tag{64}$$

on which

$$\frac{D}{Dt}[z - \zeta(x, y, t)] = 0,$$

or

$$w = \frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y}. \tag{65}$$

This is the kinematic condition at the free surface, relating w to ζ. At the free surface,

$$p = -T\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{66}$$

in which T is the surface tension and R_1 and R_2 are the principal radii of the free surface. The dynamic boundary condition is obtained from (61) and (66), and is

$$-\frac{T}{\rho}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{\partial\phi}{\partial t}+\frac{1}{2}(u^2+v^2+w^2)+(g+a)\zeta=0. \qquad (67)$$

Consider the wave motion of a liquid layer of depth h, in a container as shown in Fig. 73. If ζ and its derivatives with respect to x and y are everywhere small, u, v, and w will be everywhere small, and squares and products in u, v, w, and ζ can be neglected. Equations (65) and (67) can then be written as

$$w=\frac{\partial\zeta}{\partial t}=\frac{\partial\phi}{\partial z} \quad \text{at} \quad z=h, \qquad (68)$$

$$-\frac{T}{\rho}\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)+\left.\frac{\partial\phi}{\partial t}\right|_{z=h}+(g+a)\zeta=0. \qquad (69)$$

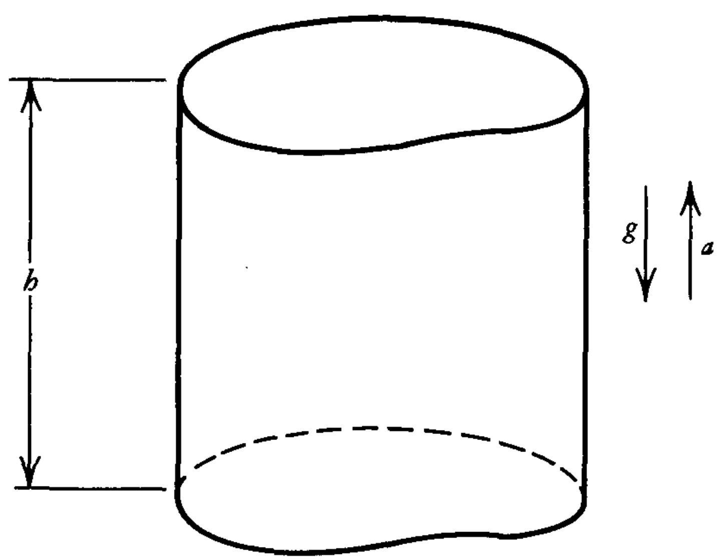


FIGURE 73. Definition sketch for instability of accelerating fluid with a free surface.

Following Benjamin and Ursell [1954], we can take

$$\zeta(x,y,t)=\sum_{0}^{\infty}a_m(t)\,S_m(x,y), \qquad (70)$$

and

$$\phi(x,y,z,t)=\sum_{1}^{\infty}\frac{da_m(t)}{dt}\,\frac{\cosh k_m z}{k_m\sinh k_m h}\,S_m(x,y)+G(t). \qquad (71)$$

$S_m(x,y)$ satisfies

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_m^2\right)S_m(x,y)=0, \qquad (72)$$

in which the k's are the eigenvalues that make $\partial S_m(x,y)/\partial n$ equal to zero on the vertical boundary, and of course depend only on the shape of the container.

Note that (63) and (68) imply that

$$\frac{\partial \zeta}{\partial n} = 0 \qquad \text{at the walls,}$$

and that (70) satisfies this condition. Also, ϕ given in (71) satisfies (62) and (63), as well as (68). (Remember $z = 0$ at the bottom of the container.)

The eigenvalue $k_0 = 0$ corresponds to $S_0(x, y) = C$. As explained by Benjamin and Ursell, $a_0(t)$ is constant, since the total volume of the liquid is constant. If the origin of ζ is taken from the mean free surface, $a_0(t) = 0$. Hence, it follows from (69) that $G(t)$ can only be a constant, which can be taken to be zero without affecting anything. With (70) and (71) substituted into (69), the result is

$$\sum_1^\infty \frac{S_m(x, y)}{k_m \tanh k_m h} \left[\frac{d^2 a_m}{dt^2} + k_m \tanh k_m h \left(\frac{k_m^2 T}{\rho} + g + a \right) a_m \right] = 0. \qquad (73)$$

Since the functions $S_m(x, y)$ are linearly independent,

$$\frac{d^2 a_m}{dt^2} + (\omega_m^2 + \omega_m'^2) a_m = 0, \qquad (74)$$

in which

$$\omega_m^2 = k_m \tanh k_m h \left(\frac{T}{\rho} k_m^2 + g \right), \qquad \omega_m'^2 = a(t)\, k_m \tanh k_m h. \qquad (75)$$

Note that $\omega_m/2\pi$ is the frequency of free oscillations and $\omega_m'/2\pi$ would be the frequency of free oscillations if surface tension were ignored and g were replaced by $a(t)$. Equation (74) is the basis for the analysis of free-surface instability due to variable acceleration.

For

$$a(t) = a_0 \cos \omega t,$$

(74) becomes [Benjamin and Ursell, 1954]

$$\frac{d^2 a_m}{d\tau^2} + (p_m - 2q_m \cos 2\tau) a_m = 0, \qquad (76)$$

if

$$p_m = 4\omega_m^2/\omega^2, \qquad q_m = 2a_0 \omega_m'^2/a(t)\omega^2,$$

and

$$\tau = \frac{1}{2}\omega t + \frac{\pi}{2}.$$

Equation (76) is Mathieu's equation in the standard form. The stability of the motion depends on the stability of a_m governed by (76). The stability chart of Mathieu's equation is given in Fig. 74.

From this figure we see that however small a_0 may be, there are always narrow regions of instability, and that there are instability regions in which free-surface oscillations are synchronous with the forcing vibrations, and there are others in which these oscillations have only half the frequency of the forcing vibrations. This explains the apparent discrepancy between the observations of Faraday [1831] and Rayleigh [1883b], who noticed half frequency, and of Matthiessen [1868, 1870], who found synchronism. But the fact that the free surface can be unstable for $a_0 \ll g$ still needs some explanation. An explanation of this instability when the free-surface oscillations have half the frequency of $a(t)$ is provided in the following paragraph.

As shown by (57), the motion relative to the container is governed by the

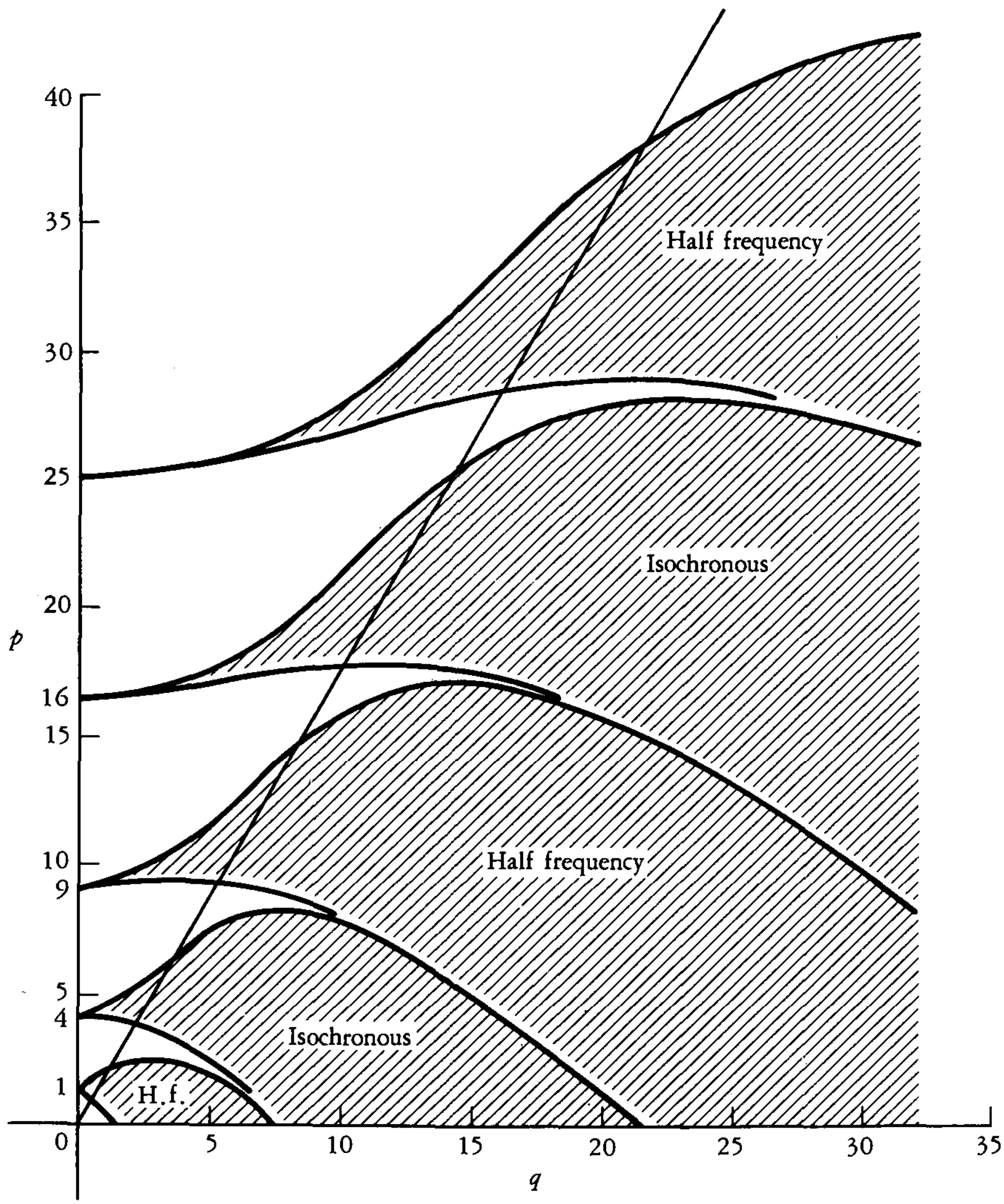


FIGURE 74. Stability diagram of the Mathieu function, after McLachlan [1947]. If $T = 0$, all modes are on a straight line for given a_0. (*Courtesy of the Clarendon Press, Oxford.*)

same Euler equations governing the motion with respect to fixed coordinates, provided g is changed to $g + a$. Thus, at least for constant a, the ordinary concept of potential energy is still applicable. Also, the total energy, kinematic and potential, relative to the container must remain constant *as long as a is constant*. Suppose that at time $t = 0$ the wave attains a maximum height, when the acceleration of the container is a constant a_1, directed upward. This acceleration is maintained until the free surface is flat, at which time all the energy of the wave motion is in the form of kinetic energy. If at this time the acceleration is suddenly changed to a constant a_2, which is less than a_1, the total energy, entirely kinetic, is unchanged, and as the next maximum wave height is attained the total energy, now entirely potential in nature, still remains unchanged; but the maximum wave height would have to be greater than the previous maximum, because the *effective* gravitational acceleration has been decreased from $g + a_1$ to $g + a_2$. This situation is described graphically in Fig. 75, and photographically in Fig. 77. The reverse is true if a_2 is greater than a_1, as shown in Fig. 76. It is also evident that the phase of

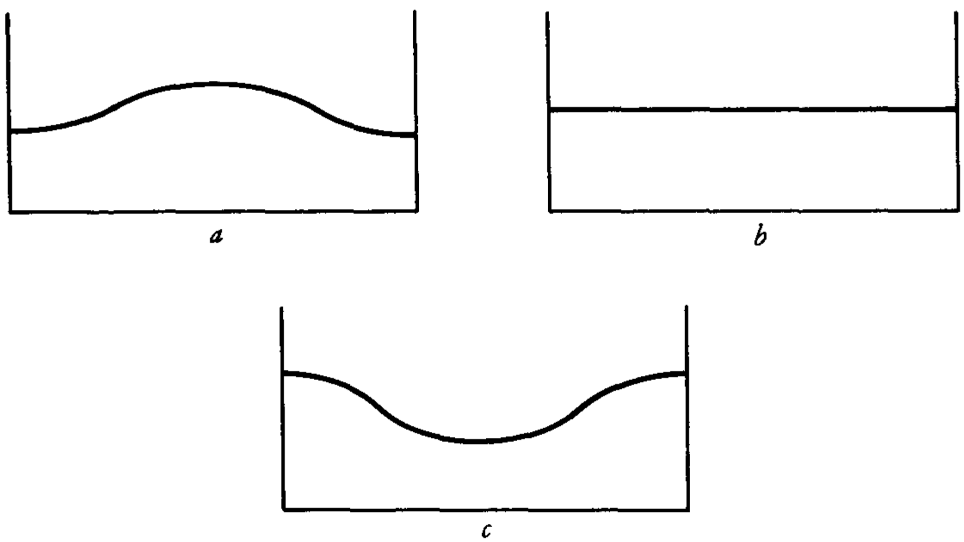


FIGURE 75. A destabilizing acceleration schedule. (a) $a = a_1$. Wave attains maximum height. (b) a changes to $a_2 < a_1$. Surface is flat. (c) $a = a_2$. Wave attains next maximum height. (*Courtesy of the Technical Association of the Pulp and Paper Industry.*)

the waves is important at the moment of change of the acceleration, whether it is decreased or increased. The instability described here explains the instability of paper-making pulp as it is carried over a Fourdrinier wire screen over the table rolls which drain the water in the pulp [Yih and Lin, 1963]. As the pulp (called stock) leaves the table rolls its acceleration changes from a downward one to a large upward one and finally to zero. Since disturbances of all phases and many wavelengths are present, the varying acceleration causes instability. It also explains in physical terms Faraday's observation of half frequency, since as the liquid accomplishes half an oscillation (Fig. 75), the vessel has already accomplished one oscillation, ready to start the next

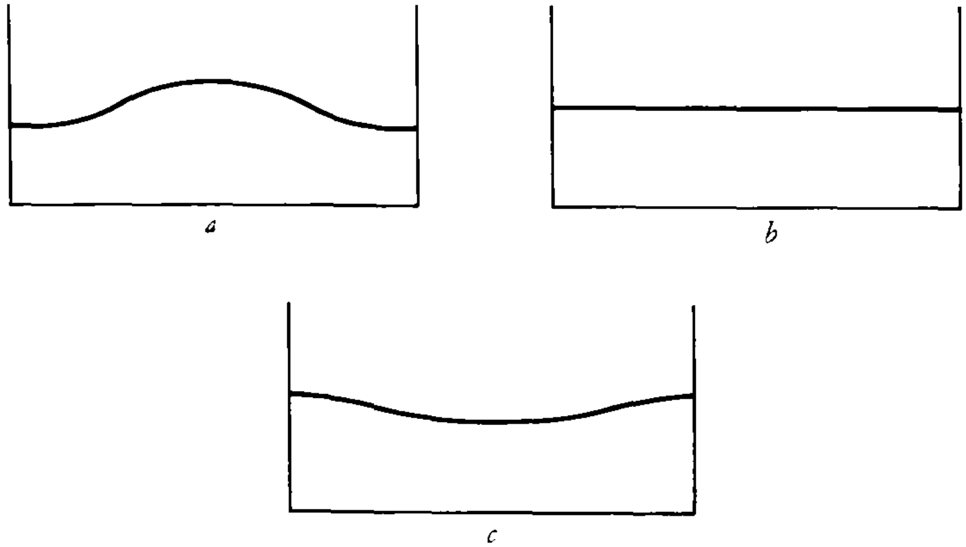


FIGURE 76. A stabilizing acceleration schedule. (a) $a = a_1$. Wave attains maximum height. (b) a changes to $a_2 > a_1$. Surface is flat. (c) $a = a_2$. Wave attains next maximum height. (*Courtesy of the Technical Association of the Pulp and Paper Industry.*)

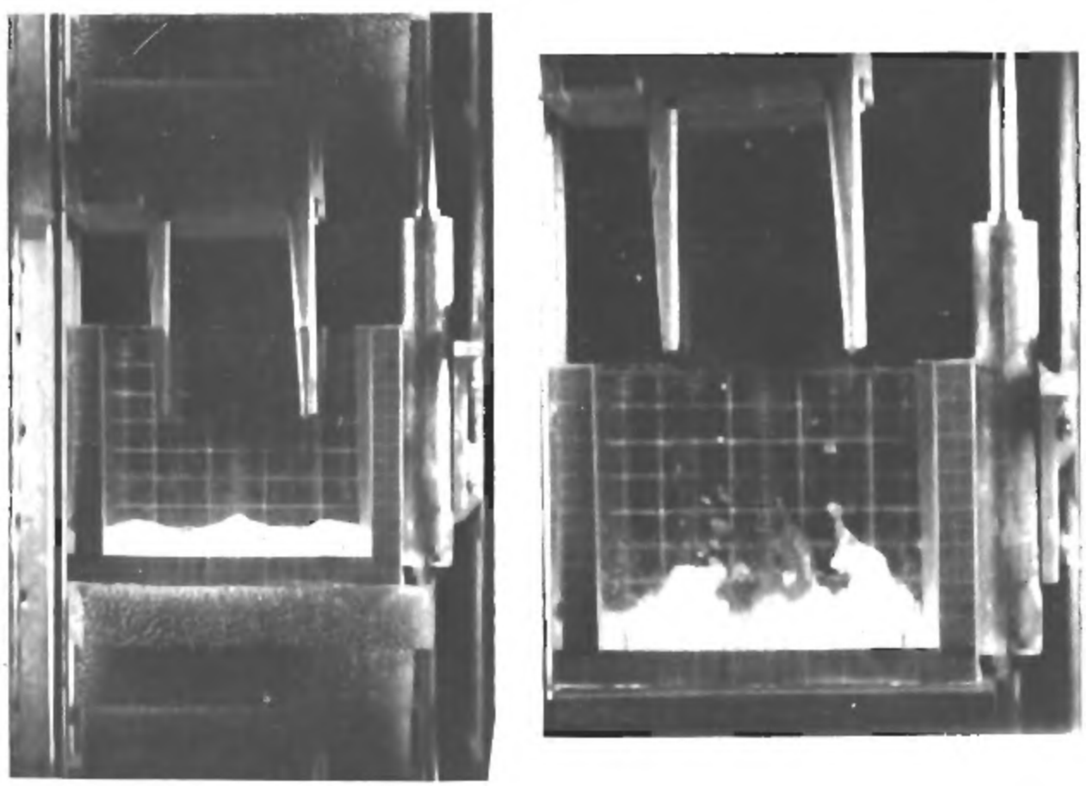


FIGURE 77. Photographs showing instability of a free surface due to varying acceleration. (Left) Before the upward acceleration was reduced. (Right) After it had been reduced. (*Courtesy of the Technical Association of the Pulp and Paper Industry.*)

cycle by changing a_2 to a_1 again. Note that the change in acceleration occurs both at extreme wave heights and at zero wave heights. The consideration given in this paragraph can be easily put into mathematical terms. In fact, the effect of wave phase at the time of change in acceleration on the growth of maximum wave height can also be taken into account rather simply in the mathematical treatment. The explanation, admittedly qualitative and incomplete, cannot replace the mathematical treatment of Benjamin and Ursell. But it at least explains in physical terms how instability can occur when the acceleration is not downward and greater in magnitude than g, and focuses attention on the *variation* of acceleration as the cause of instability.

9. STABILITY OF ROTATING FLUIDS

In the remarkable paper of Taylor [1923], the stability of an incompressible fluid between concentric cylinders rotating at different angular speeds is discussed in detail. This is not only a very successful paper, because the experiments obtained by Taylor agree beautifully with his analytical results, but also a very influential and fruitful one, as evidenced by the amount of research inspired by it in the forty years since its publication. The details of Taylor's analysis and experiments will not be presented here, for we are primarily interested in presenting the broad features of the analogy between the dynamics of rotating fluids and that of stratified fluids. For the details, the reader is referred to Taylor's work and related later work by others as described or referred to in Lin's book on hydrodynamic stability [1955]. It will only be pointed out here that, for the special case of small spacing between the cylinders and nearly equal angular speeds, the differential system governing stability is entirely the same as that governing the stability of a layer of fluid heated from below, the essentials of which have been presented in Section 4, Chapter 4. (If the temperature decreases upward, and there is no concentration gradient and hence no mass diffusion, the formulation given there is exactly applicable to the stability of a fluid layer heated from below.) In the thermal-convection problem, the critical value of the Rayleigh number as defined by the third equation in (4.60) is 1708. The corresponding parameter for the stability of fluid between rotating cylinders is the Taylor number

$$T = \frac{4\omega_1(r_2 - r_1)^3(\omega_1 r_1^2 - \omega_2 r_2^2)}{\nu^2(r_2 + r_1)},$$

in which r_1 and ω_1 are the radius and angular speed of the inner cylinder, r_2 and ω_2 are those of the outer cylinder, and ν is the kinematic viscosity. In the particular case mentioned, T assumes the form

$$\frac{2\omega_1 d^3(\omega_1 r_1^2 - \omega_2 r_2^2)}{r_1 \nu^2}.$$

in which $d = r_2 - r_1$, and its critical value is 1708. In fact, another analogy is provided by the instability of an electrically conducting fluid between concentric, stationary, and nonconducting cylinders, with a current J along their common axis and a current density j_0 in the fluid. If J and j_0 are counted positive along a direction of the axis, μ is used here (and in Chapter 7 only) as the magnetic permeability, and η is the magnetic diffusivity, the stability parameter for small spacing d is

$$\frac{J(\pi j_0 r_1^2 - J)}{\pi \rho \nu \eta}\left(\frac{d}{r_1}\right)^4,$$

and its critical value is 1708 [see Yih, 1959b]. In the case of thermal instability, the destabilizing force is buoyancy. In the case of instability of revolving fluids, it is mainly the so-called centrifugal force, although Coriolis acceleration does play a part. In the case of hydromagnetic instability, it is the electromagnetic centripetal force.

The effects of diffusivities in gravitational instability also have their counterparts in rotational instability and in hydromagnetic instability. Before going into this, special cases characterized by zero viscosity, zero thermal and mass diffusivities, and zero magnetic diffusivity will be discussed. For these *nondiffusive* cases, the *necessary and sufficient* conditions for stability are as follows:

(a) For gravitational stability, the density should decrease upward.

(b) For axisymmetric cellular motion in rotating fluids, the quantity $\rho\Gamma^2$, Γ being the circulation, should be nondecreasing with the radial distance r. This is the result of Rayleigh [1916], explained physically by von Kármán [see Lin, 1955, p. 49]. This condition has been shown by Synge [1938] to hold good even for viscous fluids.

(c) For axisymmetric motion in a conducting fluid carrying an electric current in the axial direction, the quantity H_θ^2/r^2 must not increase with r, H_θ being the θ-component of the magnetic field. This condition was found by Michael [1954], and rediscovered by Yih [1959b], who was not aware of Michael's work. However, Yih also gave it for the special case of a *viscous* fluid of *finite* conductivity between concentric nonconducting and stationary cylinders, whereas Michael's work is only for inviscid fluids of infinite conductivity. For an inviscid, infinitely conducting fluid carrying an axial current and revolving about an axis, Michael [1954] found that, with V denoting circumferential velocity, if

$$r^{-3} D(r^2V^2) - \frac{\mu}{4\pi\rho} r D\left(\frac{H_\theta^2}{r^2}\right) \geq 0, \qquad D = \frac{d}{dr}, \tag{77}$$

the fluid is stable, otherwise unstable. This condition was quoted by Chandrasekhar [1961, p. 396] and by Howard and Gupta [1962]. Howard and Gupta also developed comparable ones for the more general cases in which either an axial velocity or an axial component of the magnetic field is included. We shall see that for a real fluid (77) and Howard and Gupta's conditions are neither necessary nor sufficient for stability.

The analogy between the three categories of instability is stengthened by these necessary and sufficient conditions for inviscid and nondiffusive fluids. However, while these conditions, given separately for a single factor of stability or instability, are entirely valid even for viscous and diffusive fluids, they are not valid if more than one factor is involved and if viscosity and diffusivity are taken into account. In Section 4 of Chapter 4, it has been shown that an apparently stable stratification can be actually unstable because of diffusivities. The analogous situation for a viscous fluid between rotating cylinders and with a thermally effected radial stratification was studied by Yih [1961b], who showed that even if $\rho\Gamma^2$ increases outward the fluid may still be unstable because of thermal diffusivity or viscosity. The physical arguments go as follows. If ρ increases and Γ^2 decreases outward, in such a way that $\rho\Gamma^2$ *increases* outward, thermal diffusivity is destabilizing, because a material ring, when displaced radially, will be somewhat harmonized thermally with its new surroundings, so that the quantity $\rho\Gamma^2$ of the displaced ring may be greater than its original value if it is displaced outward, or less if displaced inward, by the action of thermal diffusivity. In either case the fluid ring will tend to move farther away from its original position, and the fluid will tend to be unstable. If, on the other hand, ρ decreases and Γ^2 increases outward in such a way that $\rho\Gamma^2$ increases outward, viscosity will, in so far as it tends to harmonize the Γ of a displaced ring with that of its new surroundings, be destabilizing. That the destabilizing effects of diffusivity and viscosity can be sufficiently great to bring about actual instability is shown in Yih's paper [1961b], to which the reader is referred for details. Yih's work shows that when both ρ and Γ vary with r, the condition $(d/dr)(\rho\Gamma^2) \geq 0$ is not sufficient for the stability of a viscous and diffusive fluid, for which it is also known not to be necessary.

Wei Lai [1961 and 1962] has shown that thermal and magnetic diffusivities can be destabilizing in a conducting fluid carrying an axial current between rotating concentric cylinders, and that the destabilizing effects can be sufficiently great to cause actual instability. This shows that condition (77) of Michael [1954] and the corresponding ones* of Howard and Gupta [1962] (for cases in which either an axial flow or an axial field is present in addition to V and H_θ) are invalid as either a necessary or a sufficient condition for stability of a real fluid.

The analogy between stratified flows and swirling flows is further enriched by Howard and Gupta's derivation [1962] of a sufficient condition for the stability of swirling flows with longitudinal velocity, corresponding to Miles' condition ($J > \frac{1}{4}$) for stratified flows, and by their derivation of a semicircle

* Equations (37) and (60) of their paper.

theorem for swirling flows, corresponding to Howard's own beautiful semicircle theorem for stratified flows. If V denotes the circumferential velocity and W the axial velocity, Howard's sufficient condition for stability is

$$r^{-3}\frac{d}{dr}(r^2V^2) \geq \frac{1}{4}W'^2,$$

and his semicircle theorem for swirling flows is: If $(d/dr)(r^2V^2) \geq 0$, the complex wave speed c for any unstable mode must lie inside the semicircle in the upper half plane which has the range of W as diameter.

The instability of a rotating liquid film with a free surface has been studied by Yih [1960d], that of a rotating liquid cylinder by Hocking and Michael [1959] and by Hocking [1960]. The instability of a liquid film (Fig. 78) inside a rotating cylindrical container has been demonstrated by Debler and Yih [1961]. All these instabilities are analogous to gravitational instability.

FIGURE 78. Rings forming in a liquid film attached to the inside wall of a rotating cylinder, as the cylinder is slowed down or suddenly stopped. (*From Debler and Yih [1962], J. Aerosp. Sci.* **29**, *364.*)

For the many results concerning stability of a fluid heated below and in rotation or in the presence of a magnetic field, the reader is referred to Chandrasekhar's book [1961], which contains an exhaustive bibliography of work on this subject done at the University of Chicago.

10. FINITE CAVITIES ATTACHED TO ACCELERATING BODIES

To bring out further the analogy between fluid flow in a field of gravity and the flow of an accelerating fluid, we shall consider the case of cavitation behind or *in front of* an accelerating body. The fluid is assumed inviscid and of constant density.

It is necessary to distinguish between the case of an ambiently accelerating fluid past a stationary body and the case of a body accelerating in an ambiently quiescent fluid, if a cavity is present. This distinction is not necessary if the body or the ambient fluid has a constant velocity, even if a cavity is present, as is well known. It is also unnecessary if the body or the ambient fluid is accelerating but no cavity is present, for the two problems are kinematically equivalent, and the pressure distributions for the two cases differ only by a linear function of x, which is measured in the direction of the acceleration. If both acceleration and cavitation are present, however, the distinction is necessary because the boundary condition on the surface of the cavity is a dynamic one, and a difference in pressure distribution will involve a difference in the geometry of flow. Since the case of the accelerating body is the more realistic one, only that case will be discussed.

Consider the two-dimensional flow caused by a body moving with velocity $-U$ in the x-direction [Yih, 1960c]. The coordinates moving with the body are denoted by (x, y), and those with respect to fixed axes momentarily coinciding with the moving axes are denoted by (x', y'). Thus,

$$x = x' + \int_0^t U\,dt, \qquad y = y'. \tag{78}$$

If the instantaneous potential function is denoted by $\Phi_1(x', y', t)$, the potential with respect to axes moving with the body is

$$\Phi(x, y, t) = \Phi_1(x', y', t) + Ux', \tag{79}$$

if the velocity is the gradient of Φ (rather than the negative of that). On the surface of the cavity, $\Psi(x, y, t)$ is constant, if Ψ is conjugate to Φ, and if the cavity is of permanent form. The dynamic boundary condition on the surface of the cavity is

$$\frac{\partial \Phi_1}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi_1}{\partial x'}\right)^2 + \left(\frac{\partial \Phi_1}{\partial y'}\right)^2\right] = F(t), \tag{80}$$

which can be written as

$$\frac{\partial \Phi}{\partial t} + ax + \frac{\Phi_x^2 + \Phi_y^2}{2} = 0, \tag{81}$$

since

$$\begin{aligned}\frac{\partial \Phi_1}{\partial t} &= \left.\frac{\partial \Phi}{\partial t}\right|_{x', y'} - ax' \\ &= \frac{\partial \Phi}{\partial t} + U\frac{\partial \Phi}{\partial x} - ax'\end{aligned} \tag{82}$$

and

$$\frac{\partial \Phi_1}{\partial x'} = \frac{\partial \Phi}{\partial x} - U, \qquad \frac{\partial \Phi_1}{\partial y'} = \frac{\partial \Phi}{\partial y}. \tag{83}$$

In (82) $a = dU/dt$. In (81) the right-hand side has been put equal to zero without loss of generality. If now

$$\Phi(x, y, t) = U f(x, y), \tag{84}$$

(81) becomes

$$a(f + x) + \frac{U^2}{2}(f_x^2 + f_y^2) = 0. \tag{85}$$

If λ is real and positive, and

$$-a + \lambda U^2 = 0,$$

$$U = \frac{U_0}{1 - U_0 \lambda t}, \tag{86}$$

in which U_0 is the value of U at $t = 0$. Equation (86) gives the time-dependence of U, corresponding to a cavity of permanent form, and is due to von Kármán [1949]. Equation (85) then becomes

$$f_x^2 + f_y^2 = -2\lambda(f + x), \tag{87}$$

or

$$\phi_x^2 + \phi_y^2 = 2\lambda^2(-\phi + \lambda x), \tag{88}$$

in which

$$f = \frac{\phi}{\lambda}, \qquad \Phi = \frac{U}{\lambda}\phi. \tag{89}$$

Equation (88) is satisfied by [Yih, 1960e] with a different sign convention

$$2\lambda \frac{dz}{dw} = H'(w) - i\left[\frac{4}{H(w) - 2w} - H'^2(w)\right]^{1/2}, \tag{90}$$

in which the accents indicate differentiation with respect to w, H' and the square root of the bracket must be real on the surface of the cavity, and H must be such that there be no singularities outside of the body and the cavity. Formula (90) corresponds to Richardson's formula—(3.122). With $H(w) = w$,

$$2\lambda \frac{dz}{dw} = 1 - i\left(\frac{4}{-w} - 1\right)^{1/2}, \tag{91}$$

which satisfies the condition that the velocity is U and in the direction of x at $w = \infty$, relative to the body. From $w = \infty$ (real) to $w = 0$, dz/dw is real, so that the streamline is the line $y = 0$, say. At $w = 0$ the streamline branches, and between $w = 0$ and $w = -4$ the streamline $\psi = 0$ (ψ is conjugate to ϕ) is described by (with $w_1 = -w$)

$$2\lambda z = -w_1 \mp i[\sqrt{w_1(4 - w_1)} - 4\cos^{-1}\sqrt{\tfrac{1}{4}w_1} + 2\pi], \tag{92}$$

in which arc cosine takes the value between 0 and $\pi/2$, and the upper and lower signs are to be taken for the lower and upper branches of the streamline, respectively. Equation (92) is a direct counterpart of (3.134), and the cavity shape is as given in Fig. 31, except it should be turned 90° clockwise. From $w = -4$ to $w = -\infty$ (real), dz/dw is again real, thus the boundary of the body is again straight, as shown in Fig. 31.

If λ is kept constant and the sign of U_0 is changed, U is still given by (86) but its direction is changed, and dU/dt keeps the same sign. Furthermore, ϕ is unchanged but the sign of Φ is changed and the flow reversed. Thus the cavity for a body moving with decreasing speed occurs in *front*, with its form exactly the same as for the case in which U_0 is positive and U is increasing. This is an interesting feature. The above discussion is for λ being positive. The case of negative λ can be similarly discussed. For negative λ and positive U_0, (86) shows that U is positive for all positive t and decreases to zero as $t \to \infty$. This is in contrast with the case of positive λ and U_0, in which U approaches infinity as t approaches $(U_0\lambda)^{-1}$.

NOTES

Section 3

1. Zeytounian [1967] considered axisymmetric flows of a rotating ideal gas. Coriolis forces are taken into account but a β-plane approach is adopted and cylindrical coordinates are used. The analysis is neither linear nor rigorously nonlinear. The author did not attempt to justify his neglect of certain nonlinear terms while retaining others. Assuming a linear vertical distribution for the temperature, he managed to give first integrals of the equations of motion and of heat balance, using the equation of continuity in the process.

Section 4

1. Jacobs [1964] studied the flow of a rotating stratified fluid bounded above by a horizontal plane and below by a bottom in the form of a long ridge on a horizontal plane. The flow far upstream is horizontal, uniform, barotropic, and normal to the length of the ridge. He showed that if the Rossby number based on the upstream velocity, the width of the ridge, and the angular velocity of rotation are small and the ridge has a small height, then lee waves are negligible. Ignoring, then, the terms giving rise to lee waves and using temperature to replace the vertical coordinate, he was able to arrive at a linear differential equation and simple boundary conditions.

His solution gives the effects of stratification on transverse flow in the direction along the ridge, and shows that when the stratification is strong the transverse flow near the bottom of the fluid upstream of the ridge has a direction opposite to that at the lee side of the ridge. Topographic effects are not confined to a region near the bottom but are felt throughout the entire depth of the fluid, as a result, no doubt, of the general rotation of the fluid.

General

1. Further results substantiating the analogy between the dynamics of stratified fluids and that of rotating fluids can be found in an article by Veronis [1970].

2. An admirable book on rotating fluids by Greenspan [1968] defines by example the terms "applied mathematics" and "excellence." It is a book that every serious student of fluid mechanics should have in his library.

Chapter 7

ANALOGY BETWEEN GRAVITATIONAL AND ELECTROMAGNETIC FORCES

1. INTRODUCTION

It is well known that the gravitational force arising from the presence of masses has a potential. Indeed the potential can be readily calculated by applying Newton's law of universal gravitation, and the work of Gauss and Green in the nineteenth century on gravitational fields has given rise to the term *potential theory*. In contrast, the body force $\mu\mathbf{j} \times \mathbf{H}$ arising from the magnetic field $\mathbf{H}$ and the electric current density $\mathbf{j}$, where μ denotes the magnetic permeability exclusively in this chapter, does not have a potential except, perhaps, in very special cases. This difference has important consequences. For instance, even if the fluid is of constant density and viscous effects are neglected, the presence of the electromagnetic force will in general invalidate Kelvin's circulation theorem, which is given by (1.57) with ρ constant, and will thus make irrotational motion impossible in general. From this point of view it would seem that phenomena in which gravitational force plays an important part and those in which electromagnetic force plays an important part would be quite different and that there would be little analogy or similarity between them.

There is, however, much similarity between them, provided the fluid is of variable density in a gravitational field, for then although the gravitational acceleration (or gravitational force per unit mass) g does have a potential the force per unit volume ρg does not, ρ being the variable density. Thus if ρ is variable the aforementioned difference between gravitational forces and electromagnetic forces does not preclude the possibility of some analogy between their effects after all. We note in this connection that the Coriolis acceleration, which is called the Coriolis force when moved to the other side of the equation of motion, also does not have a potential. And yet, as shown in Chapter 6, there is often an analogy between the dynamics of rotating fluids and that of stratified fluids. Indeed, we have mentioned, in passing, at the appropriate places in Chapter 6 that the behavior of a conducting fluid in magnetic fields is often analogous to that of a rotating or a stratified fluid. In what follows we shall now bring out in detail some important similarities between flows of a fluid stratified in density and magnetohydrodynamic flows. The development is far from exhaustive and is intended only to put these similarities in relief, in the belief that similarity among seemingly unrelated phenomena always aids understanding and therefore gives satisfaction.

2. THE EQUATIONS GOVERNING FLUID MOTION IN A MAGNETIC FIELD

The symbols used in Chapter 1 will retain their meanings here. In order not to complicate things unnecessarily, the density ρ and the viscosity will be assumed constant here. The vector equation of motion is

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\mathbf{grad}\,(p + \rho\Omega) + \mu\mathbf{j}\times\mathbf{H} + \rho\nu\nabla\cdot\nabla\mathbf{v}, \qquad (1)$$

where $\mathbf{v}$ is the velocity vector, ν the kinematic viscosity, p the pressure, and Ω the gravitational potential. The current density $\mathbf{j}$ is related to the magnetic field $\mathbf{H}$ by

$$4\pi\mathbf{j} = \mathbf{curl}\,\mathbf{H}. \qquad (2)$$

Furthermore,

$$\mathbf{j} = \sigma(\mathbf{E} + \mu\mathbf{v}\times\mathbf{H}), \qquad (3)$$

where σ is the conductivity and $\mathbf{E}$ is the intensity of the electric field, which is related to $\mathbf{H}$ by

$$\mathbf{curl}\,\mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}. \qquad (4)$$

The equation of continuity for the magnetic field is

$$\mathbf{div}\,\mathbf{H} = 0. \qquad (5)$$

which is consistent with (4).

Using (2), we can write (1) as

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\mathbf{grad}\,P - \mathbf{grad}\,\frac{\mu H^2}{8\pi} + \mathbf{div}\,\frac{\mu\mathbf{H}\mathbf{H}}{4\pi} + \rho\nu\nabla\cdot\nabla\mathbf{v}, \quad (6)$$

where

$$P = p + \rho\Omega, \qquad H = |\mathbf{H}|.$$

Eliminating **E** between (3) and (4) and using (2), we have

$$\frac{\partial \mathbf{H}}{\partial t} = \mathbf{curl}\,(\mathbf{v} \times \mathbf{H}) + \eta \nabla \cdot \nabla \mathbf{H}, \tag{7}$$

where

$$\eta = (4\pi\mu\sigma)^{-1} \tag{8}$$

is the magnetic diffusivity.

In Cartesian coordinates x_1, x_2, and x_3, (6) has the form

$$\rho\left(\frac{\partial}{\partial t} + u_\alpha \frac{\partial}{\partial x}\right)u_i = -\frac{\partial}{\partial x_i} P - \frac{\partial}{\partial x_i}\frac{\mu H^2}{8\pi} + \frac{\mu}{4\pi} H_\alpha \frac{\partial H_i}{\partial x_\alpha} + \rho\nu\nabla^2 u_i, \tag{9}$$

if u_1, u_2, and u_3 are the components of **v**, H_1, H_2, and H_3 are the components of **H**, and (7) has the form

$$\left(\frac{\partial}{\partial t} + u_\alpha \frac{\partial}{\partial x_\alpha}\right)H_i = H_\alpha \frac{\partial u_i}{\partial x_\alpha} + \eta\nabla^2 H_i, \tag{10}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_\alpha \partial x_\alpha}$$

is the Laplacian.

In cylindrical coordinates r, θ, and z, we shall denote the corresponding velocity components by u, v, and w and the corresponding magnetic field strength by H_r, H_θ, and H_z. Then, written out in full, (6) becomes

$$\rho\left(\frac{Du}{D\tau} - \frac{v^2}{r}\right) = -\frac{\partial \chi}{\partial r} + \rho\nu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right) + \frac{\mu}{4\pi}\left(\frac{\mathscr{D}H_r}{\mathscr{D}\tau} - \frac{H_\theta^2}{r}\right), \tag{11}$$

$$\rho\left(\frac{Dv}{D\tau} + \frac{uv}{r}\right) = -\frac{1}{r}\frac{\partial \chi}{\partial \theta} + \rho\nu\left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right) + \frac{\mu}{4\pi}\left(\frac{\mathscr{D}H_\theta}{\mathscr{D}\tau} + \frac{H_r H_\theta}{r}\right), \tag{12}$$

$$\rho\frac{Dw}{D\tau} = -\frac{\partial \chi}{\partial z} + \rho\nu\nabla^2 w + \frac{\mu}{4\pi}\frac{\mathscr{D}H_z}{\mathscr{D}\tau}, \tag{13}$$

in which

$$\chi = p + \frac{\mu|\mathbf{H}|^2}{8\pi} + \rho\Omega, \tag{14}$$

$$\frac{\mathscr{D}}{\mathscr{D}\tau} \equiv H_r\frac{\partial}{\partial r} + \frac{H_\theta}{r}\frac{\partial}{\partial \theta} + H_z\frac{\partial}{\partial z}, \tag{15}$$

$$\frac{D}{D\tau} \equiv \frac{\partial}{\partial \tau} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}, \tag{16}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \tag{17}$$

and (7) becomes

$$\frac{DH_r}{D\tau} = \frac{\mathscr{D}u}{\mathscr{D}\tau} + \eta\left(\nabla^2 H_r - \frac{H_r}{r^2} - \frac{2}{r^2}\frac{\partial H_\theta}{\partial\theta}\right), \tag{18}$$

$$\frac{DH_\theta}{D\tau} + \frac{vH_r}{r} = \frac{\mathscr{D}v}{\mathscr{D}\tau} + \frac{H_\theta u}{r} + \eta\left(\nabla^2 H_\theta - \frac{H_\theta}{r^2} + \frac{2}{r^2}\frac{\partial H_r}{\partial\theta}\right), \tag{19}$$

$$\frac{DH_z}{D\tau} = \frac{\mathscr{D}w}{\mathscr{D}\tau} + \eta\nabla^2 H_z. \tag{20}$$

As to (5), its forms in Cartesian and cylindrical coordinates are well known.

3. THE STIFFENING EFFECT OF MAGNETIC LINES

If the effects of magnetic diffusion are negligible (which implies that the magnetic Reynolds number based on η is large), the last term in (7) may be neglected except in boundary layers, and then (7) has exactly the same form as the vorticity equation for homogeneous inviscid fluids. (See (1.50) with ρ constant, which is directly comparable with (10) with the last term omitted.) It is well known that vortex lines in an inviscid fluid of constant density move with the fluid; i.e., once a material line is a vortex line it will forever be a vortex line. Hence if magnetic diffusivity is negligible magnetic lines also move with the (conducting) fluid. To use Alfvén's expression [Cowling, 1957, pp. 5–6], the magnetic lines are "frozen" into the fluid. Yih [1959e] has shown that in the presence of a strong uniform magnetic field weak steady motions are independent of the distance in the direction of the magnetic field, as if the magnetic lines had stiffened the fluid and constrained it to move in that way. This stiffening effect is directly analogous to the gravitational effect of density or entropy stratification.

Consider a uniform magnetic field H_0 acting in the x-direction, and let a weak steady motion take place in this field. If the perturbation magnetic field is denoted by $\mathbf{h}$,

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}. \tag{21}$$

Since the motion is steady and weak, the left-hand side of (9) can be neglected. If, in addition, the effects of viscosity are neglected (which implies that the Reynolds number is large even though the motion is weak in relation to the magnetic forces present), the three components of (9) are, when written out in full,

$$\frac{\partial P}{\partial x} = 0, \qquad (22$$

$$\frac{\partial P}{\partial y} = \frac{H_0}{4\pi}\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right), \qquad (23$$

$$\frac{\partial P}{\partial z} = \frac{H_0}{4\pi}\left(\frac{\partial h_z}{\partial x} - \frac{\partial h_x}{\partial z}\right), \tag{24}$$

in which x, y, and z denote Cartesian coordinates and h_x, h_y, and h_z are the components of **h** in these coordinates. Cross differentiation of (22) to (24) gives, upon use of (5) in one step,

$$\frac{\partial}{\partial x} \operatorname{\mathbf{curl}} \mathbf{h} = 0, \tag{25}$$

or, in view of (2),

$$\frac{\partial}{\partial x} \mathbf{j} = 0. \tag{26}$$

Similarly, the left-hand side of (10) can be neglected, and if the effect of magnetic diffusion is negligible (which implies that the magnetic Reynolds number is large), the last term in (10) can be neglected, and then (10) gives, upon dropping the nonlinear terms in the term following the equality sign.

$$\frac{\partial}{\partial x} \mathbf{v} = 0. \tag{27}$$

Equations (26) and (27) say that the electric-current density and the velocity are independent of x. The flow may be in all three dimensions, but the velocity components depend only on y and z. That is, mathematically there are only two dimensions.

On account of (27), a material line once parallel to the x-axis will always be so. If then $\mathbf{h} = 0$ far upstream (say at $y = -\infty$), so that all magnetic lines there are parallel to the x-axis, everywhere downstream the magnetic lines will be parallel to the x-axis, since under the assumption of negligible diffusive effects magnetic lines move with the fluid. Hence

$$h_y = h_z = 0,$$

and from (5)

$$\frac{\partial h_x}{\partial x} = 0. \tag{28}$$

However, h_x does depend on y and z, since j_y and j_z are not zero, as can be seen from (3).

Thus the magnetic lines of the field $\mathbf{H}_0$ seem to have stiffened the fluid and constrained it to move two-dimensionally in the physical or at least the mathematical sense, even if the fluid flows past a three-dimensional body. This is directly analogous to the gravity effects of density variation treated in Section 3 of Chapter 1, or the stiffening effect of vortex lines mentioned in Section 4 of Chapter 6.

For circular magnetic fields (due to electric currents) Yih [1959e] showed that steady weak flows are independent of the azimuth θ.

4. CENTRIPETAL WAVES

Consider an electrically conducting fluid filling the annular space between two coaxial cylinders of radius r_1 and r_2 ($> r_1$). The z-axis will be taken to be the axis of the cylinders.

Now consider a circular magnetic field described by

$$\bar{H}_r = 0 = \bar{H}_z, \qquad \bar{H}_\theta = \bar{H}_\theta(r). \tag{29}$$

The axial current in the fluid is given by

$$j_0 = \frac{1}{4\mu r}\frac{d}{dr}(r\bar{H}_\theta). \tag{29a}$$

If the three components of the disturbance to the magnetic field are denoted by h_r, h_θ, and h_z, and if viscous and (magnetically) diffusive effects are neglected, the equations of motion and of the magnetic field are obtained directly from Eqs. (11) to (20):

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial r} + \frac{\mu}{4\pi}\frac{\partial}{\partial r}(\bar{H}_\theta h_\theta) - \frac{\mu}{4\pi r}\bar{H}_\theta h_\theta, \tag{30}$$

$$\rho\frac{\partial v}{\partial t} = \mu j_0 h_r, \tag{31}$$

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p'}{\partial z} + \frac{\mu}{4\pi}\frac{\partial}{\partial z}(\bar{H}_\theta h_\theta), \tag{32}$$

$$\frac{\partial h_\theta}{\partial t} = -r\frac{d}{dr}\left(\frac{\bar{H}_\theta}{r}\right)u, \qquad \frac{\partial h_r}{\partial t} = 0, \qquad \frac{\partial h_z}{\partial t} = 0. \tag{33}$$

From the last two equations in (33), we can take h_r and h_z to be zero if the motion has started from rest or if the motion is periodic in time—which is the case for wave motion. Then from (31) we see that v vanishes also. Eliminating p' between (30) and (32), we have

$$\rho\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) = -\frac{\mu}{4\pi r}\bar{H}_\theta\frac{\partial h_\theta}{\partial z}. \tag{34}$$

Since, for axisymmetric motion

$$\frac{\partial(ru)}{\partial r} + \frac{\partial(rw)}{\partial z} = 0, \tag{35}$$

elimination of w between (34) and (35) gives

$$\rho\frac{\partial}{\partial t}Lu = -\frac{\mu}{4\pi}q\frac{\partial^2}{\partial z^2}h_\theta, \tag{36}$$

where

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2},$$

$$q = \frac{\overline{H}_0}{r}. \tag{37}$$

Elimination of h_0 between (33) and (37) gives, finally,

$$\rho \frac{\partial^2}{\partial t^2} Lu = \frac{4}{4\pi} rqq' u_{zz}, \tag{38}$$

where $q' = dq/dr$, and the subscript z indicates partial differentiation. The boundary conditions on u are

$$u = 0 \quad \text{at} \quad r = r_1 \quad \text{and} \quad r = r_2. \tag{39}$$

Let

$$u = U(r, z)e^{-i\sigma t}.$$

Then (38) becomes

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \left(1 + \frac{\mu}{4\pi\rho\sigma^2} rqq'\right)\frac{\partial^2}{\partial z^2}\right]U = 0. \tag{40}$$

This equation is of the hyperbolic type or the elliptic type according as

$$\frac{\mu}{4\pi\rho\sigma^2} rqq' < \quad \text{or} \quad > -1.$$

If wave motion is possible, σ is real. But wave motion is only possible if (40) is of the hyperbolic type. Therefore we conclude that wave motion is impossible if

$$qq' > 0, \tag{41}$$

that is, q^2 increases outward. We shall see later that (41) is indeed the condition for instability if viscous and diffusive effects are neglected.

If we further assume

$$U(r, z) = f(r)e^{ikz},$$

(40) becomes

$$f'' + \frac{1}{r}f' - \left(\frac{1}{r^2} + k^2 G\right)f = 0, \tag{42}$$

where

$$G = 1 + \frac{\mu}{4\pi\rho\sigma^2} rqq', \tag{43}$$

with the boundary conditions

$$f(r_1) = 0 = f(r_2). \tag{44}$$

Equation (42) can be put into the self-adjoint form

$$(rf')' - \left(\frac{1}{r^2} + k^2 G\right) rf = 0, \tag{45}$$

from which it can be seen, by the Sturm-Liouville theory presented in Chapter 2, that (44) cannot possibly be satisfied if σ is real and (41) holds, which is in agreement with what was said in the last paragraph. On the other hand, if

$$qq' < 0 \tag{46}$$

we see from (45) that according to the Sturm-Liouville theory there are infinitely many values of σ^2 which will allow (44) to be satisfied. That is, not only is wave motion possible but there are infinitely many modes. Indeed this statement holds if (46) is satisfied only in part of the fluid—but in that case the fluid is unstable.

We now point out the similarities between small-amplitude motions of a fluid stratified in density and those of a conducting fluid with a circular magnetic field. Equation (40) is analogous to (2.168) or (2.171). Equation (45) is analogous to (2.31) when we recall that $c = \sigma/k$ in (2.31). Finally (46) is analogous to $\bar{\rho}' < 0$, where $\bar{\rho}$ is the mean density that appeared in (2.31). It is also analogous to $d\Gamma^2/dr > 0$, Γ being the circulation, which is the condition for stability of a revolving fluid, mentioned in Section 9 of Chapter 6. The system consisting of (45) and (44) is a Sturm-Liouville system, like the system consisting of (2.31) and (2.33) or that comprising (6.28) and (6.29), which governs centrifugal waves in a rotating fluid.

Wave motion in a circular magnetic field discussed here is due to the body force $\mu\mathbf{j} \times \mathbf{H}$, which in the undisturbed state of the case under consideration is $\mu j_0 H_\theta$, pointing *toward* the axis. Since we call the waves in a rotating fluid centrifugal waves because of the centrifugal body force (which is the centripetal acceleration moved to the other side of the equation), when the body force (for the primary state) is centripetal, as is the case here, it is logical to call the corresponding waves centripetal waves. This nomenclature, of course, ignores the perturbation of the body force, as indeed the nomenclature "centrifugal waves" ignores the Coriolis force, although it also plays a crucial role there. The two names are used to point out an interesting distinction and for convenience of reference.

It is evident from (43)–(45) that wave motion is impossible if

$$\sigma^2 > \left(-\frac{\mu}{4\pi\rho} rqq'\right)_{\max} \equiv N^2. \tag{47}$$

N^2 corresponds to $(-g\bar{\rho}'/\bar{\rho})_{\max}$ for a fluid stratified in density, as can be seen upon comparing (1.170) with (45). If $qq' < 0$, N is real, and is an upper and, like the Brunt-Väisälä frequency $(-g\bar{\rho}'/\bar{\rho})^{1/2}_{\max}$, unattainable bound for the frequency σ.

5. EQUATIONS GOVERNING FINITE-AMPLITUDE AXISYMMETRIC MOTION OF A CONDUCTING FLUID

We shall now consider steady, axisymmetric motions of an infinitely conducting, inviscid, and incompressible fluid, of finite amplitude. Cylindrical coordinates will be used and the notation in Section 2 for variables in these coordinates will be retained. Adopting the simplifications

$$(f, g, h) = (\mu/4\pi\rho)^{1/2}(H_r, H_\theta, H_z), \tag{48}$$

we can obtain from the equation of continuity of the magnetic field

$$\frac{\partial(rf)}{\partial r} + \frac{\partial(rh)}{\partial z} = 0$$

the results

$$fr = -\Lambda_z, \qquad hr = \Lambda_r, \tag{49}$$

in which Λ is "stream function" for the magnetic field. From the equation of continuity of the fluid we have the familiar

$$ur = -\psi_z, \qquad wr = \psi_r, \tag{50}$$

in which ψ is the Stokes stream function. In (49) and (50) subscripts indicate partial differentiation. From these two equations we obtain directly that

$$\Lambda = \Lambda(\psi). \tag{51}$$

which shows that projections of the streamlines and magnetic lines coincide in the $r - z$ plane. For an infinitely conducting fluid $\eta = 0$, so that (7) becomes, for steady flows,

$$\textbf{curl}\,(\mathbf{v} \times \mathbf{H}) = 0, \tag{52}$$

the second component of which is, in the notation defined by (48),

$$\frac{\partial}{\partial r}(vf - ug) + \frac{\partial}{\partial z}(vh - wg) = 0. \tag{53}$$

From (50), (51), and (53), we have

$$\left(u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}\right)\left(\frac{g - \Lambda' v}{r}\right) = 0,$$

from which, upon using (50) again, we have [Long, 1960]

$$\frac{g - \Lambda' v}{r} = K(\psi), \tag{54}$$

in which the prime on Λ indicates differentiation with respect to ψ. Furthermore, for steady flows (12), upon neglecting the effects of viscosity, can be put in the form

$$\left(u \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right)(vr - \Lambda' gr) = 0,$$

which gives

$$vr - \Lambda' gr = L(\psi). \tag{55}$$

Equations (54) and (55) give

$$(vr)^2 - (gr)^2 = A - Br^4, \tag{56}$$

where

$$A = \frac{L^2}{1 - \Lambda'^2}, \qquad B = \frac{K^2}{1 - \Lambda'^2}. \tag{57}$$

Eliminating χ between (11) and (13), we have

$$\frac{\partial}{\partial z}\left[-\frac{v}{r}\frac{\partial}{\partial r}(vr) + \frac{g}{r}\frac{\partial}{\partial r}(gr) + w(u_z - w_r) - h(f_z - h_r)\right]$$
$$- \frac{\partial}{\partial r}\left[-\frac{v}{r}\frac{\partial}{\partial z}(vr) + \frac{g}{r}\frac{\partial}{\partial z}(gr) - u(u_z - w_r) + f(f_z - h_r)\right] = 0, \tag{58}$$

or, upon division by r,

$$\frac{D}{Dt}\left[\frac{(u_z - w_r) - \Lambda'(f_z - h_r)}{r}\right] - \frac{1}{r^4}\frac{\partial}{\partial z}[(vr)^2 - (gr)^2] = 0, \tag{59}$$

where

$$\frac{D}{Dt} = u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}.$$

Now, upon using (56) and (57),

$$-\frac{1}{r^4}\frac{\partial}{\partial z}[(vr)^2 - (gr)^2] = u\left(\frac{A'}{r^3} - B'r\right) = \left(\frac{A'}{r^3} - B'r\right)\frac{Dr}{Dt}$$
$$= -\frac{D}{Dt}\left(\frac{A'}{2r^2} + \frac{B'r^2}{2}\right), \tag{60}$$

since

$$\frac{D\psi}{Dt} = 0.$$

Using (60), we obtain finally

$$-\frac{u_z - w_r - \Lambda'(f_z - h_r)}{r} + \frac{A'}{2r^2} + \frac{B'r^2}{2} = M(\psi). \tag{61}$$

or, writing everything in terms of ψ,

$$\psi_{zz} + \psi_{rr} - \frac{1}{r}\psi_r - \frac{\Lambda'\Lambda''}{(1-\Lambda'^2)}[(\psi_z)^2 + (\psi_r)^2] + \frac{A'}{2(1-\Lambda'^2)} + \frac{B'r^4}{2(1-\Lambda'^2)} = \frac{Mr^2}{(1-\Lambda'^2)}, \tag{62}$$

Equation (62), as well as the intermediate equations (54), (55), and (56), are due to Long [1960]. Equation (62) is analogous to (6.50) for rotating fluids. It is also analogous to (3.11) for a stratified fluid, first derived by Dubreil-Jacotin [1935] and rediscovered by Long [1953b]. But, unlike (3.11), (62) is not a rediscovery, for, as far as this writer is aware, Long was the first to find it. Some of the many intermediate equations are far from obvious, and the first arrival at (62) must have been a delight and a triumph for Long.

Long gave two exactly linear cases for (62). In the first the flow and the magnetic field are both axial at $z = \infty$, so that the magnetic lines and the streamlines coincide, and the flow is irrotational, although the pressure is modified by the magnetic field. This case is therefore trivial. In his second case the magnetic field is axial at infinite z, but at infinite z one has a uniform axial flow combined with a solid-body rotation, and the exactly linear form of (62) is just (6.52), with the ω^2 in (6.52) divided by

$$1 - h_0^2 w_0^{-2},$$

h_0 being the h at infinity and w_0 the w at infinity. Thus the effect of the axial magnetic field is merely to modify the effective angular velocity. Yih [1965b] gave the solution for another linear case of (64), corresponding to an axial magnetic field plus a circular magnetic field due to an axial electric current at infinite z, as well as an axial flow plus a rigid-body rotation there. The flow patterns are similar to those shown in Fig. 9, depending on the values of the parameters involved.

6. SIMPLIFICATION OF LONG'S EQUATION

Equation (62) is rather complicated. Yih [1965b] made the transformation

$$d\Psi = (1 - \Lambda'^2)^{1/2}\, d\psi, \tag{63}$$

which reduces (62), after it has been multiplied by $(1 - \Lambda'^2)^{1/2}$, to

$$\Psi_{zz} + \Psi_{rr} - \frac{1}{r}\Psi_r + \frac{1}{2}\frac{dA}{d\Psi} + \frac{1}{2}\frac{dB}{d\Psi}r^4 = N(\Psi)r^2, \tag{64}$$

in which

$$N = M(1 - \Lambda'^2)^{-1/2}.$$

Equation (64) [Yih, 1965b] is analogous to (3.10) for stratified fluids. That it is considerably simpler than (62) is evident. Indeed, (62) allows *all* the exactly linear cases of (64) to be obtained—simply by putting the last three terms in (62) in linear forms of ψ. In this regard, too, it is similar to (3.10).

7. EQUATIONS GOVERNING THE MOTION OF AN INCOMPRESSIBLE STRATIFIED AND CONDUCTING FLUID

Consider first two-dimensional flows in the x-z plane, with the velocity components denoted by u and w and the stream function denoted by ψ. The z-axis is vertical. Let the components H_x, H_y, and H_z of the magnetic field be independent of y. Then there exists a two-dimensional stream function Λ for the magnetic field, in terms of which

$$\left(\frac{\mu}{4\pi\rho_0}\right)^{1/2} H_x = \Lambda_z, \qquad \left(\frac{\mu}{4\pi\rho_0}\right)^{1/2} H_z = -\Lambda_x,$$

ρ_0 being a (constant) reference density. Yih [1975b] showed that if there is a stagnation point in the flow then

$$\Lambda = \Lambda(\psi).$$

Making the transformation

$$(u', v') = E(u, v), \qquad \text{or} \quad d\psi' = E\,d\psi,$$

where

$$E \equiv \left(\frac{\rho}{\rho_0} - \Lambda'\right)^{1/2},$$

Yih [1975b] again obtained (3.10) for the modified stream function ψ'.

For axisymmetric flows with (or without) swirl and axisymmetric magnetic fields, if we make the transformation

$$d\Psi = E\,d\psi,$$

where ψ is now Stokes' stream function, with the Λ in E defined above standing for the same Λ as in (49), we obtain (64), but with the additional term

$$\frac{gzr^2}{\rho_0}\frac{d\rho}{d\psi'}$$

on its left-hand side, r and z being the same as in (64), and g being the gravitational acceleration.

8. RING VORTICES GENERATED ELECTROMAGNETICALLY

In Section 9 of Chapter 6 the stability of Couette flow (the flow of an incompressible viscous fluid in the annulus between two concentric rotating cylinders) was briefly mentioned. When Couette flow is unstable ring vortices are formed, called Taylor vortices after Sir Geoffrey Taylor [1923b] for his remarkable study. Can similar ring vortices be created in a fluid at rest? This question was answered in the affirmative by Yih [1959b], if electromagnetic forces are allowed.

Now imagine an electric current J along the z-axis which gives rise to a circular magnetic field equal to $2J/r$, according to (2). If, in addition, there is an electric current of density j_0 passing between the cylinders in the axial direction, the corresponding circular magnetic field is $2\pi j_0(r^2 - r_1^2)/r$. The total magnetic field is given by

$$\bar{H}_r = \bar{H}_z = 0, \qquad \bar{H}_\theta = 2\pi\left(\frac{J'}{\pi r} + j_0 r\right), \qquad J' = J - \pi j_0 r_1^2, \tag{65}$$

where the bar is used to indicate that the field is the mean field, or primary field, in the absence of disturbances.

A disturbance of the stationary fluid will give rise to small velocity components (u, v, w) and a deviation from the equilibrium magnetic field denoted by (h_r, h_θ, h_z). The total magnetic field is then given by

$$h_r, \qquad 2\pi\left(\frac{J'}{\pi r} + j_0 r\right) + h_\theta, \qquad h_z.$$

If these are substituted in the equations of motion and the equations of magnetic diffusion and all quadratic terms in u, v, w, and the h's are neglected, and if axisymmetry is assumed, the following linearized equations are obtained:

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial \chi'}{\partial r} + \rho\nu\left(\nabla^2 u - \frac{u}{r^2}\right) - \mu\left(\frac{J'}{\pi r^2} + j_0\right)h_\theta, \tag{66}$$

$$\rho\frac{\partial v}{\partial t} = \rho\nu\left(\nabla^2 v - \frac{v}{r^2}\right) + \mu j_0 h_r, \tag{67}$$

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial \chi'}{\partial z} + \rho\nu\nabla^2 w, \tag{68}$$

$$\frac{\partial h_r}{\partial t} = \eta\left(\nabla^2 h_r - \frac{h_r}{r^2}\right), \tag{69}$$

$$\frac{\partial h_\theta}{\partial t} = \frac{4J'}{r^2}u + \eta\left(\nabla^2 h_\theta - \frac{h_\theta}{r^2}\right), \tag{70}$$

$$\frac{\partial h_z}{\partial t} = \eta\nabla^2 h_z. \tag{71}$$

in which

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \qquad \chi' = p' + \frac{\mu}{4\pi}\bar{H}_\theta h_\theta,$$

with p' denoting the pressure perturbation. The equation of continuity is still (35), since compressibility can be neglected.

From the form of Eqs. (69) and (71) we can conclude that h_r and h_z will be damped out if they are not initially everywhere zero [see Yih, 1959c]. Then from Eq. (67) we conclude further that v will also be damped out. The differential equations to be dealt with are then (66), (68), and (70). Eliminating χ' from (66) and (68), we have

$$\rho\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right) = \rho\nu\left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right) + \mu\left(\frac{J'}{\pi r^2} + j_0\right)\frac{\partial h_\theta}{\partial z}. \quad (72)$$

We define dimensionless variables

$$(r', z') = \left(\frac{r}{r_1}, \frac{z}{r_1}\right), \qquad t' = \frac{t\nu}{r_1^2},$$

$$(u_1, w_1) = \left(\frac{ur_1}{\nu}, \frac{wr_1}{\nu}\right), \qquad h_1 = \frac{h_\theta}{j_0 r_1},$$

where r_1 and r_2 are the radii of the inner and outer cylinders. Then, on dropping the primes on t, r, and z, we can write Eqs. (72) and (70) as

$$\frac{\partial}{\partial t}\left(\frac{\partial w_1}{\partial r} - \frac{\partial u_1}{\partial z}\right) = \left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{\partial w_1}{\partial r} - \frac{\partial u_1}{\partial z}\right) + A\left(\frac{B}{r^2} + 1\right)\frac{\partial h_1}{\partial z}, \quad (73)$$

$$\frac{\partial h_1}{\partial t} = \frac{\eta}{\nu}\left(\nabla^2 - \frac{1}{r^2}\right)h_1 + \frac{4\pi B}{r^2}u_1, \quad (74)$$

in which

$$A = \frac{\mu j_0^2 r_1^4}{\rho \nu^2}, \qquad B = \frac{J'}{\pi j_0 r_1^2}. \quad (75)$$

Following Taylor, we can make the following substitutions:

$$(u_1, w_1, h_1) = [U(r)\cos mz, W(r)\sin mz, h(r)\cos mz]e^{\sigma t},$$

in which m is the wave number for the z-direction. The equation of continuity then becomes

$$rU' + U + mrW = 0, \quad (76)$$

and (74) and (73) become

$$(L - m^2 - \sigma)(L - m^2)U = -m^2 A\left(\frac{B}{r^2} + 1\right)h, \quad (77)$$

$$\left(L - m^2 - \frac{\sigma\nu}{\eta}\right)h = -\frac{4\pi B\nu}{\eta r^2}U, \quad (78)$$

in which

$$L = D\left(\frac{1}{r} D(r \ )\right), \qquad D = \frac{d}{dr}.$$

With

$$U = -m^2 Af \qquad \text{and} \qquad N = \frac{4\pi ABv}{\eta},$$

Eqs. (77) and (78) can be written as

$$(L - m^2 - \sigma)(L - m^2)f = \left(\frac{B}{r^2} + 1\right)h, \tag{79}$$

$$\left(L - m^2 - \frac{\sigma v}{\eta}\right)h = m^2 N \frac{f}{r^2}. \tag{80}$$

The boundary conditions corresponding to

$$U = W = 0 \qquad \text{at} \quad r = 1 \quad \text{and} \quad r = \frac{r_2}{r_1} = \alpha$$

are

$$f = Df = 0 \qquad \text{at} \quad r = 1 \quad \text{and} \quad r = \alpha. \tag{81}$$

The simplest realistic boundary condition for h is that which corresponds to zero electrical conductivity of the walls, that is, to $j_r = 0$. But

$$j_r = \frac{1}{r}\frac{\partial h_z}{\partial \theta} - \frac{\partial h_\theta}{\partial z} = -\frac{\partial h_\theta}{\partial z} \qquad \text{(for axisymmetry)},$$

and so the boundary condition for h is

$$h = 0 \qquad \text{at} \quad r = 1 \quad \text{and} \quad r = \alpha \tag{82}$$

for nonconducting walls. The task is to find the relationship between N and m for a given value of B from the differential system consisting of Eqs. (79) to (82). That it is very similar to the differential system governing the stability of Couette flows [Taylor, 1923b] is obvious.

8.1. *Sufficient Condition for Stability*

A sufficient condition for stability can be given on physical grounds, in the manner of Rayleigh [1916b], or on mathematical grounds, in the manner of Synge [1938]. The physical proof relies upon the fact that for a magnetically nondiffusive fluid ($\eta = 0$) the lines of force move with the fluid. The proof for this well known fact is identical with that for vorticity lines in an inviscid fluid (Lamb, 1945, p. 204) and is available elsewhere. In the preceding subsection it has been shown that H_r and H_z (or h_r and h_z) will be damped out. In a discussion of the sufficient condition for stability it can thus be assumed that only H_θ is different from zero. Since the total magnetic flux round a thin material ring of fluid is constant, and since by continuity the volume of this ring must be constant, so that its cross section varies inversely as its radius r', H_θ must be equal to qr' (r' dimensional), with q as the constant (for the ring as it moves) of proportionality. The body force per unit volume in the r-direction, which is in general [Yih, 1959c]

$$\frac{\mu}{4\pi}\left(-\frac{1}{2}\frac{\partial|\mathbf{H}|^2}{\partial r} + H_r\frac{\partial H_r}{\partial r} + \frac{H_\theta}{r}\frac{\partial H_r}{\partial \theta} + H_z\frac{\partial H_r}{\partial z} - \frac{H_\theta^2}{r}\right),$$

can in the present discussion be written as

$$-\frac{\mu}{4\pi}H_\theta\left(\frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r}\right).$$

Now imagine the thin material ring with a magnetic flux in the θ-direction to be instantaneously situated at the postion indicated by r'. The body force corresponding to the term

$$-\frac{\mu}{4\pi}H_\theta\frac{\partial H_\theta}{\partial r}$$

has the potential $\mu H_\theta^2/8\pi$. The other part of the body force is

$$-\frac{\mu}{4\pi}\frac{H_\theta^2}{r}.$$

Since the instantaneous value for r is r' for the thin ring, and since $H_\theta = qr'$, this latter part can be written as

$$-\frac{\mu}{4\pi}q^2r'.$$

The potential energy due to this part of the centripetal force is $\mu q^2r'^2/8\pi$. The potential energy due to the entire electromagnetic body force is then $\mu q^2r'^2/4\pi$, which increases with r' for any particular value of q.

Returning to the distribution of the mean field $\bar{H}_\theta$ specified earlier, we can imagine the fluid in its mean configuration to be composed of thin shells, each with a different value for q given by

$$q = \frac{\bar{H}_\theta}{r} = 2\pi\left(\frac{J'}{\pi r^2} + j_0\right).$$

Since a higher value of q^2 corresponds to a "heavier" fluid, and since stability will ensue if a "heavier" fluid shell occupies a position of lower potential energy (hence smaller r), one concludes that a sufficient condition for stability is that the quantity

$$\left(\frac{J'}{\pi r^2} + j_0\right)^2$$

does not increase outwards—a situation which is possible only if

$$J \leq -\pi j_0(r_2^2 - r_1^2) \qquad \text{or} \quad J \geq j_0 r_1^2.$$

The sufficient condition for stability, $dq^2/dr \leq 0$, is analogous to Rayleigh's criterion for the stability of revolving fluids. It was first found by Michael [1954] for inviscid fluids of infinite conductivity and was independently found by Yih [1959b], who proved it for viscous fluids of finite conductivity. His proof follows.

For a mathematical proof of the same result for viscous fluids of finite electric conductivity one turns to the dimensionless equations (79) to (82). Multiplying (79) by $r\bar{f}$ ($\bar{f}$ being the complex conjugate of f) and integrating (by parts if necessary) with respect to r between 1 and α, we have, upon utilization of the boundary conditions on f,

$$I_2 + (2m^2 + \sigma)I_1 + m^2(m^2 + \sigma)I_0 = \int_1^\alpha \left(\frac{B}{r^2} + 1\right) rh\bar{f}\, dr, \tag{83}$$

in which

$$I_0 = \int_1^\alpha r|f|^2\, dr, \qquad I_1 = \int_1^\alpha \frac{1}{r}|Drf|^2\, dr, \qquad I_2 = \int_1^\alpha r|Lf|^2\, dr.$$

Similarly, by multiplying Eq. (80) by $r\bar{h}$ and $r^3\bar{h}$ and integrating, one obtains, respectively,

$$H_1 + \left(m^2 + \frac{\nu\sigma}{\eta}\right)H_0 = -m^2N\int_1^\alpha \frac{f\bar{h}}{r}\, dr, \tag{84}$$

$$H_3 + \left(m^2 + \frac{\nu\sigma}{\eta}\right)H_2 + 2\int_1^\alpha r\bar{h}D(rh)\, dr = -m^2N\int_1^\alpha rf\bar{h}\, dr, \tag{85}$$

in which

$$H_0 = \int_1^\alpha r|h|^2\, dr, \qquad H_1 = \int_1^\alpha \frac{1}{r}|Drh|^2\, dr, \qquad H_2 = \int_1^\alpha r^3|h|^2\, dr,$$

$$H_3 = \int_1^\alpha r|Drh|^2 dr.$$

Equations (83) to (85) can be suitably combined to eliminate the integrals on the right-hand sides. The result is

$$m^2N(I_2 + 2m^2I_1 + m^4I_0) + B(H_1 + m^2H_0) + H_3 + m^2H_2)$$
$$- 2\int_1^\alpha r\bar{h}D(rh)\,dr + \sigma\left(m^2NI_1 + m^4NI_0 + \frac{B\nu}{\eta}H_0 + \frac{\nu}{\eta}H_2\right) = 0. \quad (86)$$

But since

$$\int_1^\alpha r\bar{h}D(rh)\,dr + \int_1^\alpha rhD(r\bar{h})\,dr = r^2h\bar{h}|_1^\alpha = 0,$$

the real part of the integral in Eq. (86) is zero. Taking the real part of (86), we have

$$m^2N(I_2 + 2m^2I_1 + m^4I_0) + B(H_1 + m^2H_0) + (H_3 + m^2H_2)$$
$$+ \sigma_r\left(m^2NI_1 + m^4NI_0 + \frac{B\nu}{\eta}H_0 + \frac{\nu}{\eta}H_2\right) = 0. \quad (87)$$

Now from the definitions of the integrals denoted by H, it is evident that

$$\alpha^2H_0 > H_2 \quad \text{and} \quad \alpha^2H_1 > H_3.$$

Thus, since N and B are of the same sign, if

$$-B \geq \alpha^2 \quad \text{or} \quad B > 0 \quad (88)$$

it follows from Eq. (87) that σ_r is negative and the fluid is stable. With the definition of B given by (75), the sufficient condition of stability is therefore again found to be

$$J \leq -\pi j_0(r_2^2 - r_1^2) \quad \text{or} \quad J \geq \pi j_0 r_1^2. \quad (89)$$

8.2. *Solution for Small Spacings*

For small spacings of the cylinders, the operator L in (70) and (71) can be replaced by D^2. If the dimensionless parameters are now redefined as

$$t = \frac{\tau\nu}{d^2}, \quad \xi = \frac{r - r_1}{d} \;(r \text{ dimensional}), \quad m = \frac{2\pi d}{\lambda},$$

in which d is $r_2 - r_1$, and λ is the wave length in the z-direction, Eqs. (70) and (71) can be replaced by

$$(D^2 - m^2 - \sigma)(D^2 - m^2)f = \left(\frac{d}{r_1}\right)^4\left[(B + 1) - \frac{2Bd}{r_1}\xi\right]h, \quad (90)$$

$$\left(D^2 - m^2 - \frac{\nu\sigma}{\eta}\right)h = m^2N\left(1 - \frac{2d}{r_1}\xi\right)f, \quad (91)$$

where D stands now for $d/d\xi$. The boundary conditions are

$$f = Df = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad 1, \tag{92}$$

$$h = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad 1. \tag{93}$$

We shall investigate the stability for the cases in which the two currents are in the same direction ($-B < 1$), so that the electromagnetic body force on the undisturbed state is contripetal. If B is not nearly equal to -1, (90) and (91) become (since d/r_1 is assumed to be very small)

$$(D^2 - m^2 - \sigma)(D^2 - m^2)f = (B + 1)\left(\frac{d}{r_1}\right)^4 h, \tag{94}$$

$$\left(D^2 - m^2 - \frac{\nu\sigma}{\eta}\right)h = m^2 N f. \tag{95}$$

Effectively the same equations with the same boundary conditions as for the problem at hand have been solved exactly by Pellew and Southwell [1940], who also proved that for neutral stability σ is zero and not purely imaginary. Comparing (94) and (95) with Pellow and Southwell's equations, which are (4.63), with the $R_1 - R$ therein replaced by $-R$, R being the Rayleigh number defined in (4.60), we find that the parameter corresponding to Pellow and Southwell's R is

$$T \equiv -(1 + B)N\left(\frac{d}{r_1}\right)^4 = \frac{\mu J(\pi j_0 r_1^2 - J)}{\pi\rho\nu\eta}\left(\frac{d}{r_1}\right)^4.$$

According to Pellow and Southwell's solution, then,

$$T = 1707.8. \tag{96}$$

If $1 + B$ is positive but of the same magnitude as d/r_1, and if the principle of exchange of stabilities is assumed, Eqs. (90) and (91) become

$$(D^2 - m^2)^2 f = (1 + B)\left(\frac{d}{r_1}\right)^4 (1 + \beta\xi)h, \tag{97}$$

$$(D^2 - m^2)h = m^2 N f. \tag{98}$$

where

$$\beta = -\frac{2dB}{r_1(1 + B)}. \tag{99}$$

The boundary conditions are still specified by (92) and (93). Solution of the differential system for three values of β by the method of Chandrasekhar [1954d] yields the corresponding critical values of T, as given in the accompanying table.

β	0.25		0.5		1.0	
m	3.12	3.13	3.12	3.13	3.12	3.13
T (1st approximation)	1524.5	1524.6	1372.1	1372.1	1143.4	1143.4
T (2nd approximation)	1524.4	1524.4	1371.6	1371.7	1142.4	1142.4
T (3rd approximation)	1518.0	1518.0	1365.9	1366.0	1327.7	1137.7

We note that the differential system consisting of (97) and (98) with (92) and (93) is in the same form as that given by Debler [1966] for the formation of Bérnard cells of convection in water near 4° C. Again there is a mathematical analogy between the formation of electromagnetically induced vortex rings and the formation of convection cells in a fluid stratified in density.

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Note: Other papers on stratified flows can be found in the Proceedings of the International Symposium on Stratified Flows held in 1972 in Novosibirsk, U.S.S.R. There are many reports on stratified flows with special relevance to hydraulic engineering and marine environment published by the Delft Hydraulic Laboratory (especially those by G. Abraham), and at the Laboratoire National d'Hydraulique at Chatou, France (especially those by A. Daubert, F. Boulot, and J. P. Benqué).

INDEX

ADDENDA

Page 266, line 17:

Delete "do not exist" and replace it by "must be singular near that boundary."

Page 268, line 17 from bottom:

The second c_{n1} should be 1.

Page 272, line 12:

The word "fronts" should be "points".

Page 272, lines 9 to 12 from the bottom:

Delete the sentence starting with line 12 from bottom. Keep next sentence.

Page 272–273:

Replace the last paragraph bridging pages 272 and 273 with the following:

In order for things to make sense, the continuations of the complex-conjugate modes across the stability boundary must be singular neutral modes near that boundary, although far from it on the stable side these may, and will, become nonsingular neutral modes. The singular neutral modes on the stability boundary must be multiple, probably with a multiplicity of two. These conclusions do not seem to have been enunciated or made clear in previous research. They seem natural on consideration of the Helmholtz problem of stability of two layers. Putting α back into (164), one sees after some reflection that requiring c to be real does not necessarily give one unique N–α curve. The N–α curve which is the stability boundary corresponds to a multiple root of (real) c.